*-lets for everybody: Define your own transform

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joint work with Nicki Holighaus

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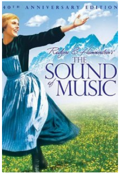
Where I am from



NuHAG: Numerical Harmonic Analysis Group, headed by Hans G. Feichtinger

2 Professors, 7 Post-Docs, 6 PhD Students

Where is the starlet?

Here is the starlet:
`http://tinyurl.com/wits-wavelets-starlet`

We will discuss: Wavelets, ERB-lets and a flexible method to construct your own *-let

Filter banks, NSGT systems

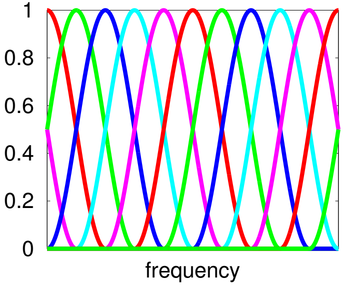
Translation: $T_x f(l) = f(l - x)$
Modulation: $M_\xi f(l) = e^{2\pi i \xi l / L} f(l)$.
Dilation: $D_a f(t) = f(ta)$ (only continuous)

- A filter bank consists of a set of filters and corresponding subsampling factors: $(g_k, b_k)_{k \in I}$
- The filter bank coefficients of a function $f \in \mathbb{C}^L$ are given by $c_{j,k} = \langle f, T_{jb_k} g_k \rangle = \langle \hat{f}, M_{-jb_k} \widehat{g_k} \rangle$, $k \in I$. In the finite discrete case $j = 0, \ldots, L/b_k - 1$, in the continuous case $j \in \mathbb{Z}$.

Gabor and Wavelet filters

Gabor filters

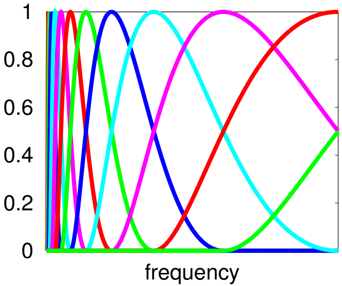
$$g_k = M_{ak} g_1, \quad \widehat{g_k} = T_{ak} \widehat{g_1}$$



Linear frequency scale

Wavelet filters

$$g_k(t) = g_1(e^{(k-1)} t)$$
$$\widehat{g_k}(t) = \widehat{g_1}(e^{-(k-1)} t)$$



Logarithmic frequency scale

Painless filter subsampling

Painless case - finite discrete

Given a set of filters and corresponding numbers of translations $(g_k, b_k)_{k=0,\dots K-1}$, where $N_k = L/b_k$. If $N_k \geq |\operatorname{supp} g_k|$, then we can achieve perfect reconstruction if there exist $0 < A, B < \infty$, such that

$$A \leq F(l) := \sum_{k=1}^{K} |\widehat{g_k}(l)|^2 \leq B.$$

Furthermore, we can reconstruct using the formula

$$f(l) = \sum_{k=0}^{K-1} \sum_{j=0}^{N_k - 1} \langle \hat{f}, M_{-jL/N_k} \widehat{g_k} \rangle g_k(l)/F(l).$$

For the continuous case: replace N_k by $1/b_k$.

Partitions of Unity - Hanning windows

It is easy to control the properties of a system of translates (Gabor filters): $\widehat{g_k} = T_{jL/N}\widehat{g_k}$.

Hanning window

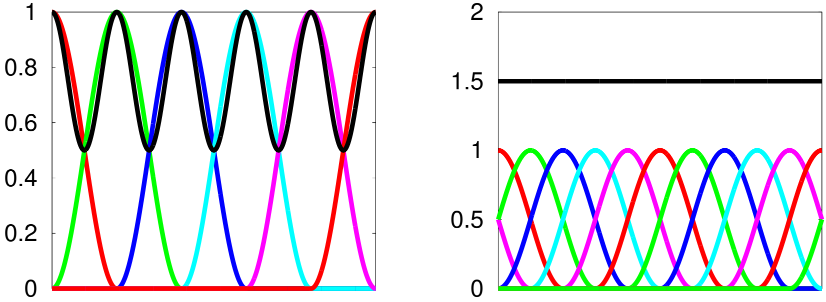
The Hanning window (raised cosine window) is defined as

$$H(t) = \frac{1}{2}(1 + \cos(\pi t))\mathbb{1}_{[-1,1]}.$$

It satisfies for some overlap constant K

$$\sum_{k\in\mathbb{Z}} |H(t - k2/K)|^2 = \text{const}\,.$$

Partitions of Unity II



Left: half overlap, Right: 3/4 overlap
Solid black line: sum of the squares of the filters

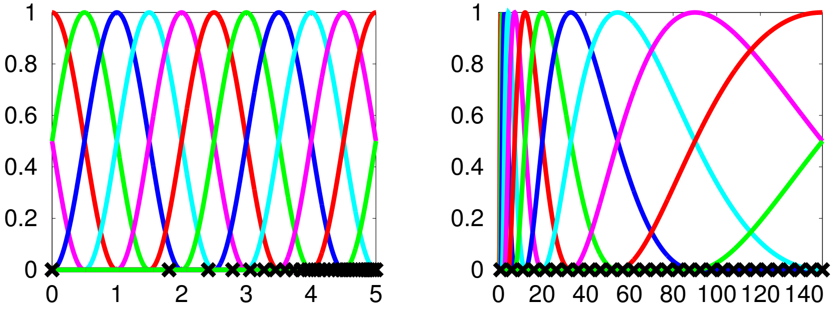
Warping - Idea

We have seen: It is easy to construct a partition of unity for translates of a function.
Generalization: non-linear evaluation of the original filters

Warping - Definition

Given a set of filters $\widehat{g_k} = T_k \widehat{g_0}$ and a warping function $F : D \to \mathbb{R}$, then we define the warped filters to be $\widehat{\varphi_k}(t) = \widehat{g_0}(F(t) + k)$.

Warping - visualization



Example for warping function $F = \log$

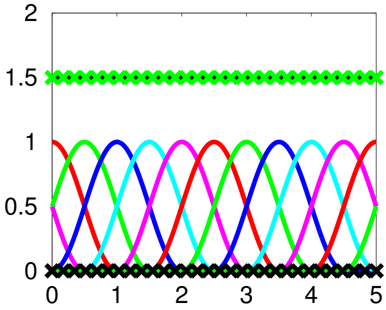
Left: original filters (translates), Right: warped filters (dilates)

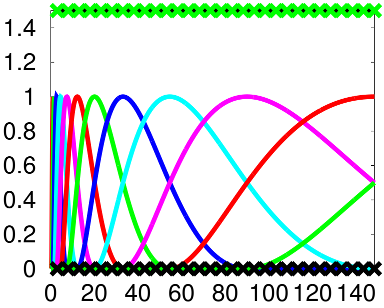
Left: $\widehat{g_k} = T_k \widehat{g_0}$, Right: $\widehat{\varphi_k}(t) = \widehat{g_0}(\log(t) + k) = \widehat{\varphi_0}(e^k t)$.

Warping and partitions of unity

The warping leaves the summation properties of the filters invariant:

$$\sum_{k=0}^{K-1} |\widehat{\varphi_k}(t)|^2 = \sum_{k=0}^{K-1} |\widehat{g}(F(t)+k)|.$$





Summary

- For filter banks the sum of the squares of the filters $F(I)$ has an important influence
- We have seen how to construct partitions of unity for a family of translates (Hanning window)
- Warping allows us to construct a filter bank following a given scale from the system of tranlates
- Up to now we have seen the logarithm as a warping function - wavelets

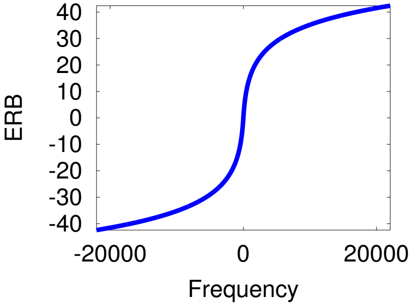
ERB warping function

ERB-scale (from Wikipedia)

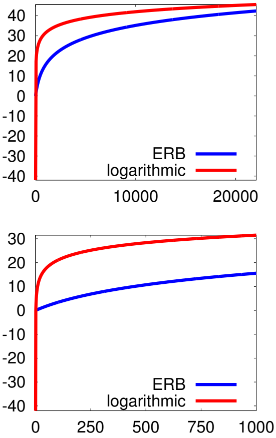
The equivalent rectangular bandwidth or ERB is a measure used in psychoacoustics, which gives an approximation to the bandwidths of the filters in human hearing, using the unrealistic but convenient simplification of modeling the filters as rectangular band-pass filters.

$$E(f) = \operatorname{sign}(f) c_1 \log(1 + |f| c_2)$$
$$c_1 = 9.2645, \;\; c_2 = 0.00437.$$

ERB-scale

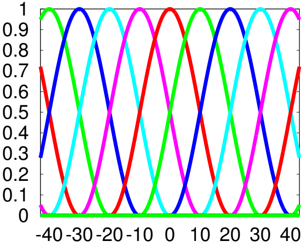


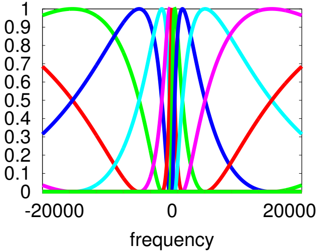
ERB
40
30
20
10
0
-10
-20
-30
-40
-20000
0
20000
Frequency



40
30
20
10
0
-10
-20
-30
-40
ERB
logarithmic
0
10000
20000
30
20
10
0
-10
-20
-30
-40
ERB
logarithmic
0
250
500
750
1000

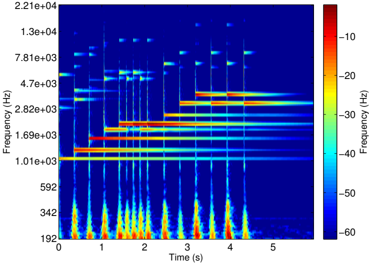
ERB-Scale





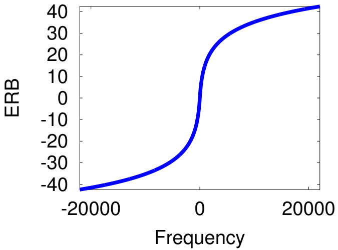
Left: system of tranlates, Right: warped system, ERB-let filters

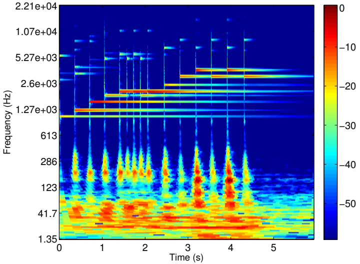
ERB-transform and Wavelet transform



Left: ERB-let transform, Right: wavelet transform

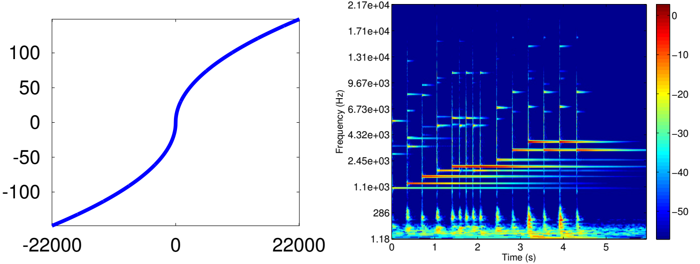
Square-Root warping function





Left: ERB warping function, Right: transform coefficients

Warping with square root



Left: warping function $F(t) = \mathrm{sign}(t)\sqrt{|t|}$, Right: transform coefficients

Thank you for your attention