Design of Gabor dual windows using convex optimization

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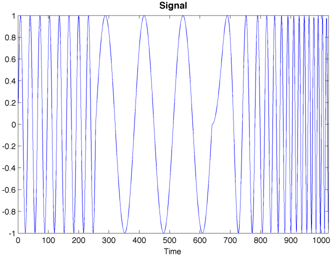
Gabor transform

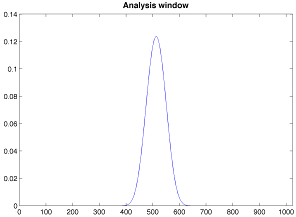
For $f, g \in \ell^2(\mathbb{Z})$, and $a, M \in \mathbb{Z}$, we define the Gabor system $\mathcal{G}(g, a, M) := \left(g_{m,n}\right)_{n\in\mathbb{Z},\ m=0,\dots,M-1}$, by

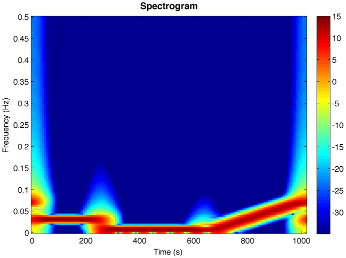
$$g_{m,n} = g[\cdot - na]e^{2\pi i m \cdot /M}, \tag{1}$$

and the Gabor transform of f by

$$(\mathbf{G}f)[m + nM] = \langle f, g_{m,n}\rangle = \sum_{l\in\mathbb{Z}} f[l]\overline{g_{m,n}[l]}. \tag{2}$$





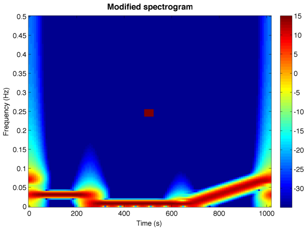


Gabor synthesis

For a coefficient sequence $c \in \ell^2(\mathbb{Z})$, Gabor synthesis is performed by applying the conjugate transpose of $\mathbf{G}$ to c.

$$f_{syn}[l] = (\mathbf{G}^* c)[l] = \sum_{m,n} c[m + nM] g[l - na] e^{2\pi i m l / M}.$$

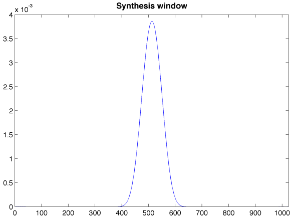
The concatenation $\mathbf{S} = \mathbf{G}^* \mathbf{G}$ is called the *frame operator*.

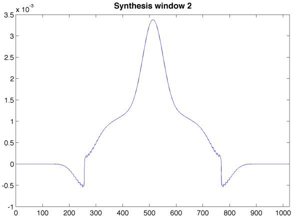


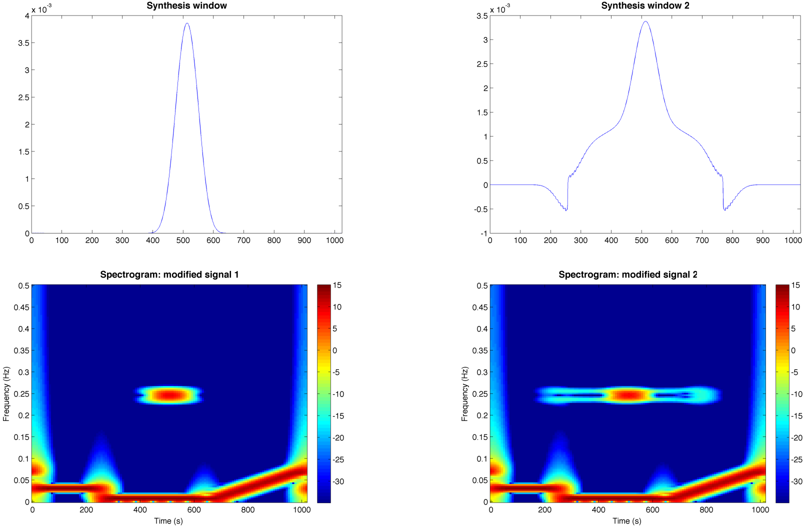
- Perfect reconstruction can be achieved using the Gabor synthesis operator using a dual window.

$$f[l] = (\mathbf{G}^* c)[l] = \sum_{m,n} \langle f, g_{m,n} \rangle \tilde{g}_{m,n}.$$

- The most common dual window is the canonical dual window defined by $\tilde{g} = \mathbf{S}^{-1} g$.
- Redundant Gabor frames admit infinitely many dual frames of the form $\mathcal{G}(h, a, M)$. They are characterized by the Wexler-Raz relations.






Synthesis window
x 10^-3
Synthesis window 2
Spectrogram: modified signal 1
Spectrogram: modified signal 2
Frequency (Hz)
Time (s)

Possible criteria for dual windows

- Compactly supported (FIR window)
- Localized in time (smooth in frequency)
- Localized in frequency (smooth in time)
- Similar to a prespecified function
- ...

The canonical dual may not always satisfy the (application specific) criteria.
Our approach: Associate mathematical objective functions and constraints with each desired criterion, and design a dual window via convex optimization.

We transform our requirements into a convex optimization problem of the form:

$$h = \underset{x \in \mathbb{R}^L}{\operatorname{argmin}} \sum_{i=1}^{K} \lambda_i f_i(x) + i_{\mathcal{C}}(x)$$

with

- $\mathcal{C}$ the set of windows satisfying our constraints. This set need to be non empty and convex.
- $f_i \in \Gamma_0(\mathbb{R}^L)$ selected regularization functions that promote certain properties.
- λ_i weights chosen in order to balance the different properties called regularizers.

Function	Effect on the signal
$\|x\|_1$	sparse representation in time
$\|\mathcal{F}x\|_1$	sparse representation in frequency
$\|\nabla x\|_2^2$	smoothen the signal in time / concentrate in frequency
$\|\nabla \mathcal{F}x\|_2^2$	smoothen in frequency / concentrate in time
$\|x\|_2^2$	spread values more evenly
$\|x - g_{sh}\|_2^2$	makes x close to g_{sh}
$\|x\|_{S_0}$	Concentrate x in time and frequency
$i_{\mathcal{C}}(x)$	force $x \in \mathcal{C}$

- Duality: satisfy the Wexler-Raz equations
- Support: be an FIR window
- Tight: produce a tight frame

The indicative function $i_{\mathcal{C}}(x)$ allows to insert the constraint into the problem.

Wexler-Raz relations [Wexler and Raz, 1990]

Two Gabor systems $\mathcal{G}(g, a, M)$, $\mathcal{G}(h, a, M)$ are dual iff

$$\frac{M}{a} \left\langle h, g[\cdot - nM] e^{2\pi i m \cdot / a} \right\rangle = \delta[n]\delta[m],$$

for $m = 0, ..., a-1,\ n \in \mathbb{Z}$.

They can also be stated as

$$\mathbf{G}_{g,M,a} h = \frac{a}{M} \delta.$$

- h is a dual window of g for a given length L if it satisfies the $\frac{L}{M}a$ Wexler-Raz equations for this interval L.
- For compactly supported dual window h only a few Wexler-Raz equations for $\ell^2(\mathbb{Z})$ are not trivially satisfied.
- The canonical dual windows is only guaranteed to be compactly supported in the painless case ($M \geq L_g$).

- The Wexler-Raz equation for tight systems,

$$\mathbf{G}^{\circ} g = \frac{a}{M}\delta,$$

do not have a convex set of solutions. $\rightarrow$ No convergence guarantee
- The projection onto tight set is given by

$$P_{\mathcal{C}_{\text{tight}}}(z) = \mathbf{S}(z)^{-\frac{1}{2}} z$$

where $\mathbf{S}(z)$ is the frame operator of $\mathcal{G}(z, a, M)$ [Janssen and Strohmer, 2002].
- Experimentally, we were still able to obtain good solutions.

- Redundancy of 8 ($M = 120$, $a = 15$)
- Fixed length of $L = 240$

$$h = \underset{x \in \mathcal{C}_{\text{dual}}}{\operatorname{argmin}} \left\{ \|\nabla \mathcal{F} x\|_2^2 + 5 \|\nabla x\|_2^2 \right\}$$

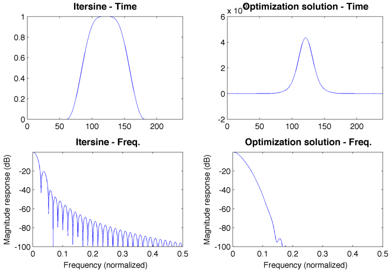


Figure : Analysis window: 'itersine' $L_g = 120$. This windows is tight. The solution of the optimization problem is close to a Gaussian window.

Example 1 (cont'd)

- Redundancy of 8 ($M = 120$, $a = 15$)
- Fixed length of $L = 240$

$$h = \underset{x \in \mathcal{C}_{\text{dual}}}{\operatorname{argmin}} \left\{ \|x - g_m\|_2^2 \right\}$$

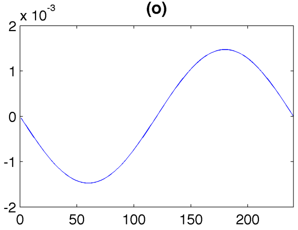


Figure : (o) Function g_m (p) Results of convex optimization problem

- Redundancy of 2 ($M = 60$, $a = 30$)
- Non painless case ($L_g = 120 > M$)
- Dual compactly supported ($L_h = 120$)

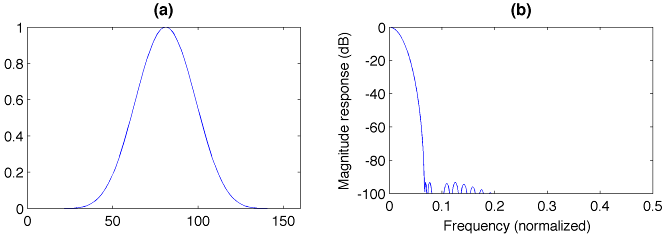


Figure : (a)(b) Synthesis window respectively in time and frequency domains.

$$h = \underset{x \in \mathcal{C}_{\text{dual}} \cap \mathcal{C}_{\text{supp}}}{\operatorname{argmin}} \left\{ \|x\|_2^2 \right\}$$

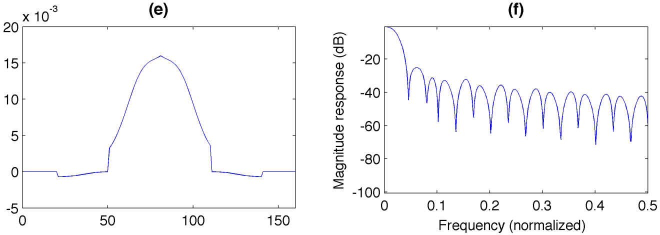


Figure : (e)(f) Synthesis window (by truncation method) respectively in time and frequency domains.

The dual window is not concentrated in frequency.

$$h = \underset{x \in \mathcal{C}_{\text{dual}} \cap \mathcal{C}_{\text{support}}}{\operatorname{argmin}} \left\{ 0.001 \|\mathcal{F}x\|_1 + 0.001 \|x\|_1 + \|\nabla \mathcal{F}x\|_2^2 + \|\nabla x\|_2^2 \right\}$$

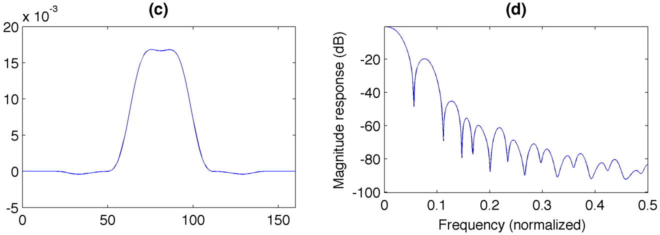


Figure : (c)(d) Synthesis window (by convex optimization) respectively in time and frequency domains

- Redundancy of 2 ($M = 60$, $a = 30$)
- Tight frame (Non convex problem)
- Dual compactly supported ($L_h = 360$)

$$h = \underset{x \in \mathcal{C}_{\text{support}} \cap \mathcal{C}_{\text{tight}}}{\operatorname{argmin}} \left\{ \|\nabla \mathcal{F} x\|_2^2 + 5 \|\nabla x\|_2^2 \right\}$$

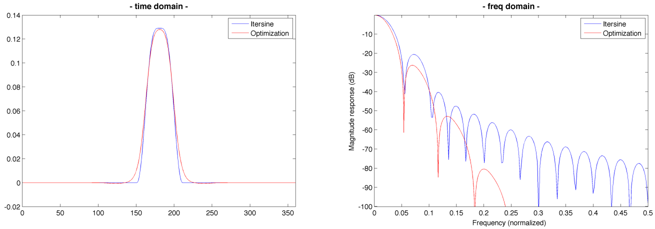


Figure : Windows respectively in time and frequency domains

- We provide a flexible method for customizing dual windows.
- The set of dual windows might be very broad. In this set, we are only limited by our own creativity.
- 3 main uses:
  - Compute windows with good properties when no other method is possible.
  - Improve existing windows.
  - Prove that an existing window is already optimal.
- The method was tested numerically and returned promising results.

Simulations were performed using the LTFAT [Søndergaard et al., 2012] and the UNLocBoX matlab toolbox. A reproducible research addendum collecting all this experiments and allowing you to design your own windows will be soon downloadable at: in `http://unlocbox.sourceforge.net/rr/gdwuco`.

- Do you have any questions?
- Thanks you for your attention

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