# Gabor and Wavelet transforms for signals defined on graphs
## An invitation to signal processing on graphs

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ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

## Swiss federal institute of technology

- 9,306 students of over 125 nationalities
- 316 laboratories, 319 faculty



Main focus: engineering, computer science, life science, biomedical engineering.

# Signal Processing Lab. 2

Prof. Pierre Vandergheynst
2 postdocs, 7 Phd Students, 1
engineer (software)

- Signal and image processing
    - 3D reconstruction, video tracking
- Sparsity, compressive sensing
    - compressive sensing for MRI data
- Optimization, inverse problems
- **Graphs and signal processing on graphs**
    - Analysis of brain data (fMRI /dMRI), graph of music, transportation networks

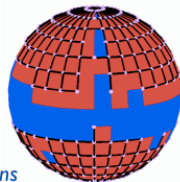From theory to applications and to start-ups.

# Graphs: models for many applications



Energy Networks

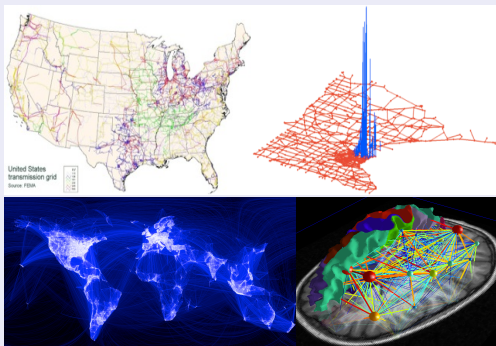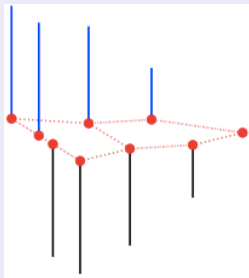Social Networks

Transportation Networks
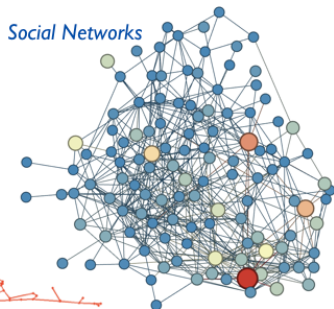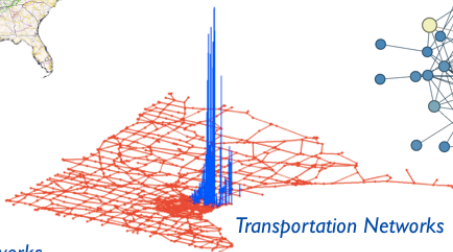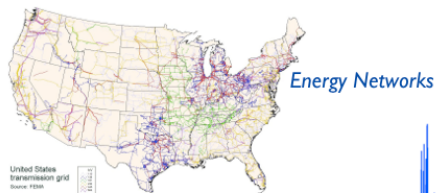
Biological Networks
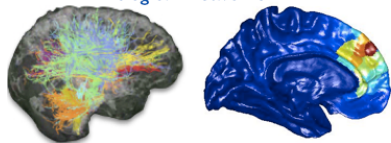
Irregular Data Domains

## Graphs: models for many applications

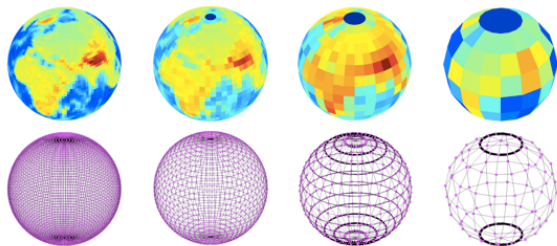Nodes $\mathcal{V}$, edges $\mathcal{E}$, weight matrix $w$.



## Data on graphs

A signal: a value or vector on each node $f : \mathcal{V} \to \mathbb{R}^N$ or $\mathbb{C}^N$. Here $N = 1$.
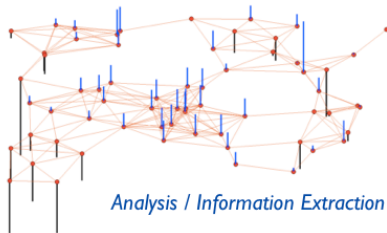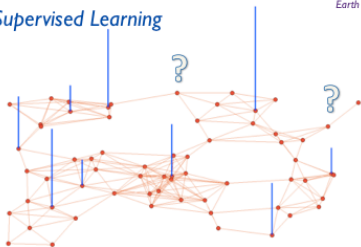
# Examples of applications

**Compression / Visualization**
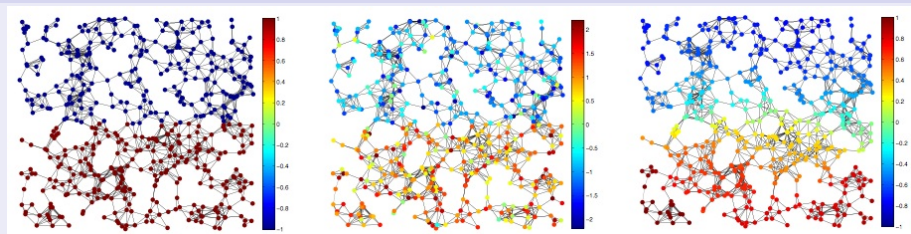


Earth data source: Frederik Simons

**Denoising**

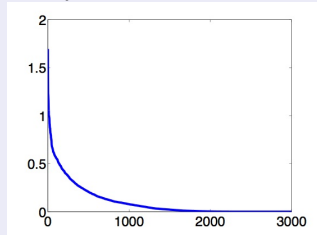**Semi-Supervised Learning**

**Analysis / Information Extraction**

# Focus: denoising and sparsity in wavelets



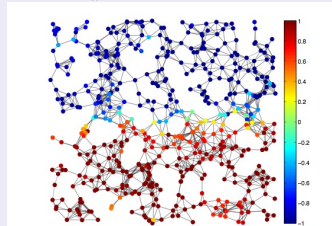(1) signal, (2) noisy signal $y$, (3) smoothing: $\operatorname{argmin}_f \|y - f\|_2^2 + \gamma \langle f, Lf \rangle$

# Wavelet denoising. Wavelet transform + thresholding

Decay of wavelet coefficients



$\operatorname{argmin}_a \|y - W^*a\|_2^2 + \gamma \|a\|_1$

## Analysis of functions, key concepts

- Smoothness, regularity of functions *on graphs*
  - The Laplacian
- Locality
- Wavelet transform
- Gabor transform

# The graph Laplacian $L$
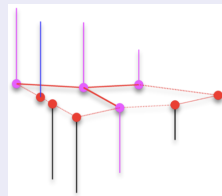
## Regularity of a function on the graph

*Smooth function: function with small variations from node to node.* Measures of the variations:
Gradient: a value for each edge $\ell^2(\mathcal{V}) \to \ell^2(\mathcal{E})$

$$\nabla f(m,n) = \sqrt{w(m,n)}[f(n) - f(m)]$$

Laplacian: a value for each node $\ell^2(\mathcal{V}) \to \ell^2(\mathcal{V})$

$$Lf(n) = \nabla^*\nabla f(n) = \sum_m w(m,n)[f(n) - f(m)]$$

## Choice

+ graph Laplacian well studied in math + used in the wavelet definition $\to L$.
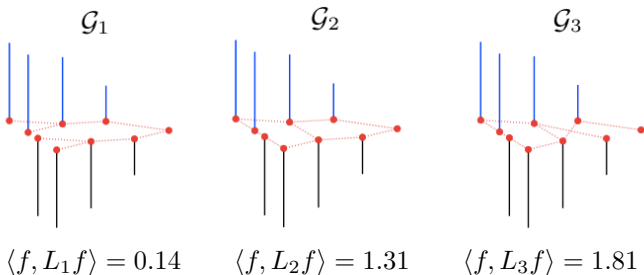
# Global regularity

Regularity based on the Laplacian.

- The *graph Laplacian quadratic form* (used for Tikhonov reg.)

$$\|f\|_L := \|\nabla f\|_2 = \sqrt{\langle f, Lf \rangle}.$$

- Sobolev regularity ($p \in \mathbb{N}$): $\|L^p f\|_2 + \|f\|_2$



$\langle f, L_1 f \rangle = 0.14$     $\langle f, L_2 f \rangle = 1.31$     $\langle f, L_3 f \rangle = 1.81$

## Wavelets (Hammond et al. ACHA, 2011)

Main challenges:

- define translation,
- define dilation.

Classical case:

- translation: convolution with a delta, multiplication in the Fourier domain,
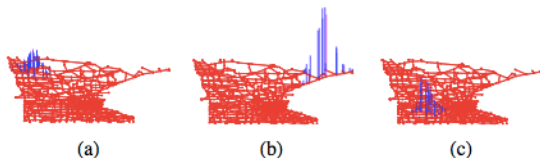- dilation: inverse dilation in the Fourier domain.

## Fourier on graphs

Fourier domain = spectral domain of the Laplacian.

- Classical case: $Le^{ikx} = -\frac{d^2}{dx^2}e^{ikx} = k^2 e^{ikx}$.
- Graph case: eigenvectors $\{u_k\}$ of $L$ = Fourier modes $Lu_k = \lambda_k u_k$.
- Graph Fourier transform:

$$\widehat{f}(k) = \langle f, u_k \rangle.$$

# Translations on the graph



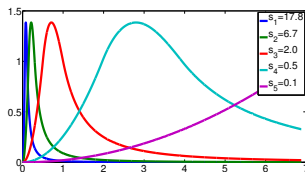   (a)           (b)           (c)

## Generalization of the convolution

- Localization around point $a$: convolution $f * \delta_a$. Multiplication in Fourier

Classical: $\qquad T_a f(x) = f(x-a) = \int_{\mathbb{R}} \widehat{f}(k) e^{-2\pi i k a} e^{2\pi i k x} dk$

Graph: $\qquad T_a f(j) = \sum_n \hat{f}(n) u_n^*(a) u_n(j)$

- We prove that it stays localized. but the shape changes as well as the $\ell^2$-norm.
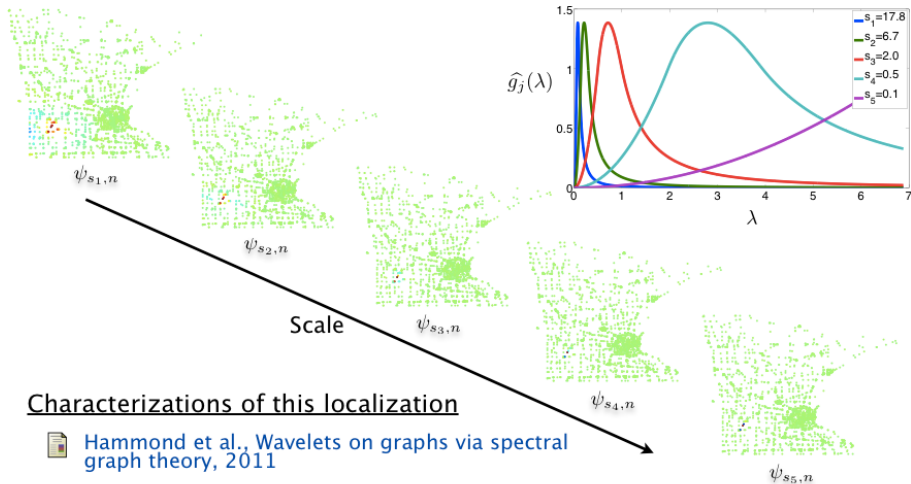
# Dilations on the graph



## Fourier domain

Dilation in the Fourier domain. *Fourier domain = a line*

1. define a continuous function $\widehat{g}$ on $\mathbb{R}_+$, "Wavelet kernel",
2. dilate it at different scales $\widehat{g_s}(\lambda) = \widehat{g}(s\lambda)$,
3. take the discretized version of $\widehat{g_s}$ on the spectrum,
4. compute the inverse Fourier transform of each $\widehat{g_s}$.

$$\psi_{s,a}(j) = \sum_n \left( \widehat{g_s}(n).u_n^*(a) \right) u_n(j)$$

# Dilations on the graph (2)



Scale

$\psi_{s_1,n}$

$\psi_{s_2,n}$

$\psi_{s_3,n}$

$\psi_{s_4,n}$

$\psi_{s_5,n}$

## Characterizations of this localization

📄 Hammond et al., Wavelets on graphs via spectral graph theory, 2011

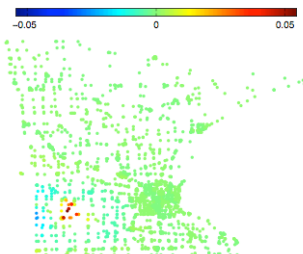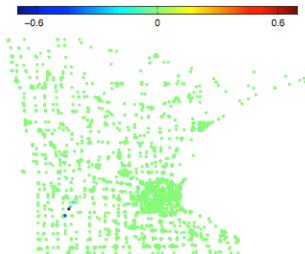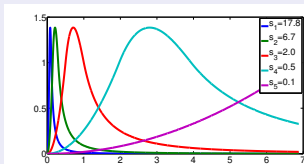📄 Shuman et al., Vertex–frequency analysis on graphs, 2013

# Wavelets (Hammond et al. ACHA, 2011)

- Eigendecomposition of the Laplacian.

- Kernel $\widehat{g}$ defined on the spectral domain. $\widehat{g} : \mathbb{R}_+ \to \mathbb{R}$ *continuous.*

- Wavelet: $\psi_{s,n}(j) = \sum_l \widehat{g}(s\lambda_l)\overline{u_l(n)}u_l(j)$.

- Wavelet transform: $Wf(s,n) = \langle f, \psi_{s,n} \rangle$.

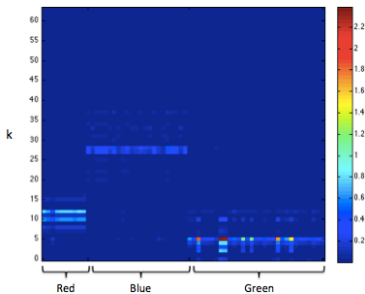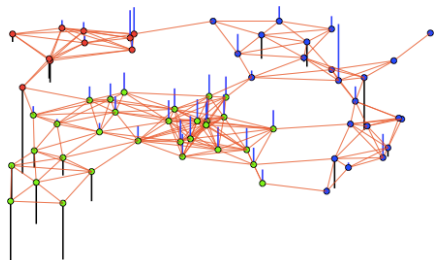- Invertible transform: add scaling function $h$ (low-pass filter).

- Admissibility cond. $\int_0^\infty \widehat{g}(x)^2 dx/x < \infty$
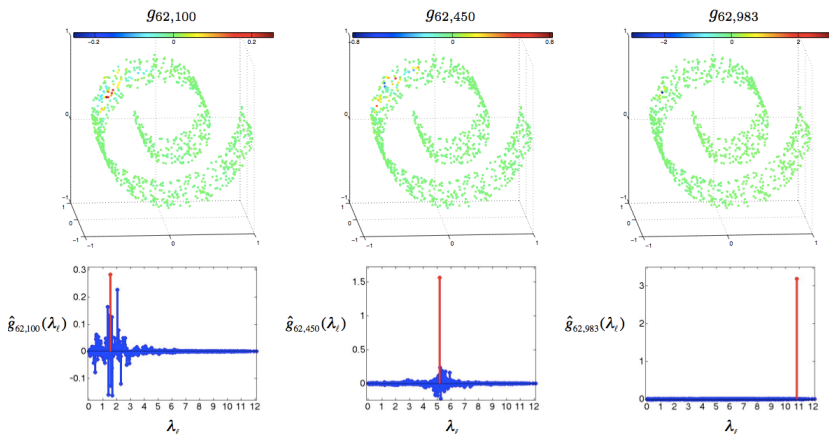
# A Gabor transform on graphs

## Ingredients

- a window,
- a translation,
- a modulation.

# Modulations on the graph

Modulation: multiplication by a Fourier mode (eigenmode of the Laplacian).

$$M_k f(i) = \sqrt{N} u_k(i) f(i).$$

# Gabor frame

The set

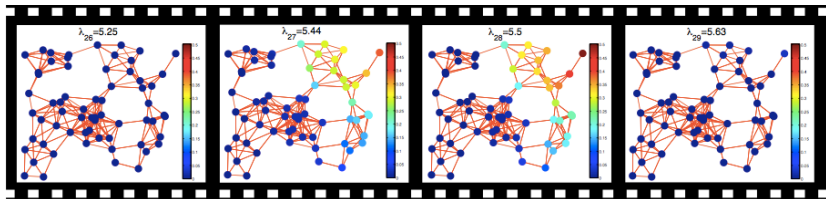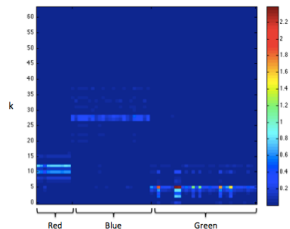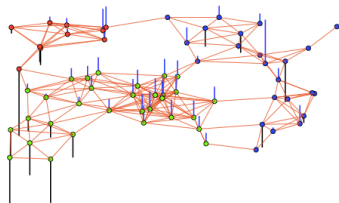$$M_b T_a g(i) = \sqrt{N} u_b(i) \sum_n \widehat{g}(n) u_n^*(a) u_n(i)$$

is a frame:

$$A\|f\|_2^2 \leq \sum_{a=1}^{N} \sum_{b=0}^{N-1} |\langle f, M_b T_a g \rangle|^2 \leq B\|f\|_2^2$$

$$0 < N|\widehat{g}(0)|^2 \leq A = \min_{a=\{1,2,...,N\}} N\|T_a g\|_2^2$$

$$B \leq \max_a N\|T_a g\|_2^2$$

Shuman, Ricaud, Vandergheynst, Vertex-frequency analysis on graphs, 2013.

# Visualization

# Conclusion

## Function behavior on the graph

- Globally smooth, slow variations of $f$,
- piecewise smooth (communities), spikes, fast variations in small regions,
- oscillations.

## Representation and sparsity

- Piecewise smooth functions *on graphs*: Sparse representation in wavelets
- (localized) oscillations: sparsity in Gabor

## Applications

- compression, compressive sensing,
- denoising,
- inpainting, label propagation...