

Beyond incoherence and beyond sparsity: compressed sensing in the real world

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EPSRC

Pioneering research
and skills

University of Cambridge, UK

Applied Functional and Harmonic Analysis Group

Head of Group

- ▶ Anders Hansen

Research Associates

- ▶ Jonathan Ben-Artzi, Bogdan Roman

PhD Students

- ▶ Alex Bastounis, Milana Gataric, Alex Jones, Clarice Poon

Interests within sampling theory

- ▶ Computations in infinite dimensions,
- ▶ Inverse Problems, Nonuniform sampling
- ▶ Compressed sensing, including applications to medical imaging (MRI and CT), seismic tomography.

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Related Group: Cambridge Image Analysis - lead by Carola Schönlieb.

- ▶ Image and video processing via PDE and variational based methods

Outline

Introduction

A new theory for compressed sensing

(Joint work with Ben Adcock, Anders Hansen and Bogdan Roman)

Compressed sensing with non-orthonormal systems

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Compressed sensing with non-orthonormal systems

A reconstruction problem in \mathbb{C}^N

- ▶ We are given two orthonormal bases

$$\Psi = \{\psi_j \in \mathbb{C}^N : j = 1, \dots, N\}, \quad \Phi = \{\phi_j \in \mathbb{C}^N : j = 1, \dots, N\}$$

- ▶ We want to **recover** $x \in \mathbb{C}^N$ where the underlying signal f is $f = \sum_{j=1}^N x_j \phi_j$.
- ▶ We are given access to **samples** $\hat{f} = (\langle f, \psi_j \rangle)_{j=1}^N$.

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We can construct a measurement matrix $U \in \mathbb{C}^{N \times N}$ with entries $u_{ij} = \langle \phi_j, \psi_i \rangle$.

$$Ux = \hat{f}$$

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$$P_{\Omega} U x = P_{\Omega} \hat{f}$$

where P_{Ω} is the projection matrix onto the index set $\Omega \subset \{1, \dots, N\}$.

Current theory

IF we have

1. Sparsity

$$|\{j : x_j \neq 0\}| = s \ll N,$$

2. Incoherence

$$\mu(U) := \max_{j,k=1,\dots,N} |\langle \phi_j, \psi_k \rangle|^2 = \mathcal{O}(N^{-1})$$

Then by choosing $s \log(N) \log(\epsilon^{-1} + 1)$ samples **uniformly at random**, x is recovered exactly with probability exceeding $1 - \epsilon$ by solving

$$\min_{\beta \in \mathbb{C}^N} \|\beta\|_1 \quad \text{subject to } P_\Omega U \beta = P_\Omega \hat{f}.$$

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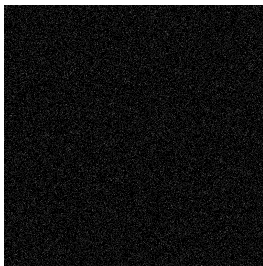
$$\min_{\beta \in \mathbb{C}^N} \|\beta\|_1 \quad \text{subject to } P_\Omega U \beta = P_\Omega \hat{f}.$$

- ▶ In MRI, we are interested in Fourier sampling with wavelet sparsity. But, the coherence between Fourier and any wavelet basis is $\mu = 1$.
- ▶ There are many important applications which are **highly coherent**. e.g. X-ray tomography, electron microscopy, reflection seismology.

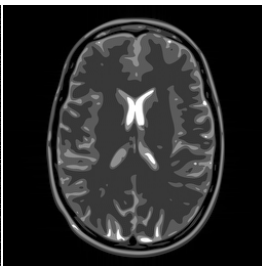
Uniform random sampling

$$\min_{\beta \in \mathbb{C}^N} \|\beta\|_1 \quad \text{subject to} \quad P_{\Omega} U_{df} V_{dw}^{-1} \beta = P_{\Omega} \hat{f}.$$

5% subsampling map
(1024x1024)



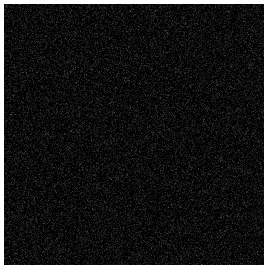
Original



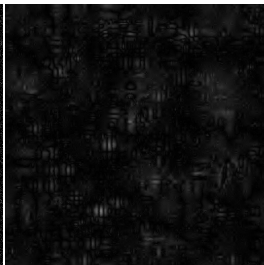
Test phantom constructed by Guerquin-Kern, Lejeune, Pruessmann, Unser, '12

Uniform random sampling

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(1024x1024)

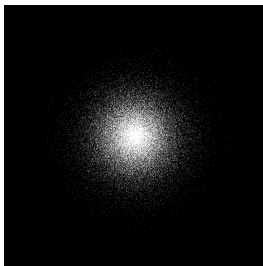


Reconstruction
(1024x1024)

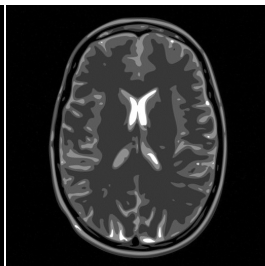


Variable density sampling

5% subsampling map
(1024x1024)



Reconstruction
(1024x1024)



- ▶ **Empirical solution** (Lustig (2008), Candès (2011), Vandergheynst (2011)): take more samples at lower Fourier frequencies and less samples at higher Fourier frequencies.
- ▶ **Q:** How can we theoretically justify this approach?

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Introduction

A new theory for compressed sensing

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Compressed sensing with non-orthonormal systems

Multi-level sampling

Divide the samples into r levels and let

- ▶ $\mathbf{N} = (N_1, \dots, N_r)$ with $1 \leq N_1 < \dots < N_r = N$,
- ▶ $\mathbf{m} = (m_1, \dots, m_r)$ with $m_k \leq N_k - N_{k-1}$, $N_0 = 0$,
- ▶ $\Omega_k \subset \{N_{k-1} + 1, \dots, N_k\}$ is chosen uniformly at random, $|\Omega_k| = m_k$,

We refer to the set

$$\Omega_{\mathbf{N}, \mathbf{m}} = \Omega_1 \cup \dots \cup \Omega_r$$

as an (\mathbf{N}, \mathbf{m}) -sampling scheme.

To understand the use of multi-level random sampling instead of uniform random sampling, we replace the standard ingredients of compressed sensing with **more realistic** assumptions:

Sparsity	→	Asymptotic sparsity
Incoherence	→	Asymptotic incoherence

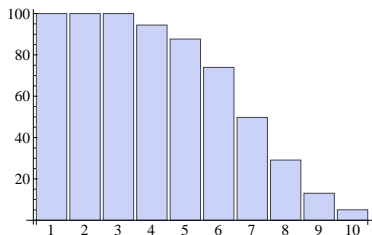
Asymptotic sparsity

Divide the reconstruction coefficients into levels and consider sparsity at each level: For $r \in \mathbb{N}$, let

- ▶ $\mathbf{M} = (M_1, \dots, M_r)$ with $1 \leq M_1 < \dots < M_r = N$,
- ▶ $\mathbf{s} = (s_1, \dots, s_r)$ with $s_k \leq M_k - M_{k-1}$, $M_0 = 0$,

We say that $x \in \mathbb{C}^N$ is **(\mathbf{M}, \mathbf{s})-sparse** if for each $k = 1, \dots, r$,

$$|\text{supp}(x) \cap \{M_{k-1} + 1, \dots, M_k\}| \leq s_k.$$



Left: test image, Right: Percentage of wavelet coefficients at each scale which are greater than 10^{-3} in magnitude.

Asymptotic incoherence

We say that $U \in \mathbb{C}^{N \times N}$ is asymptotically incoherent if for large N ,

$$\mu(P_K^\perp U), \mu(UP_K^\perp) = \mathcal{O}(K^{-1})$$

where P_K is the projection matrix onto the index set $\{1, \dots, K\}$.

For any wavelet reconstruction basis with Fourier samples

$$\mu(U) = 1, \quad \mu(P_K^\perp U), \mu(UP_K^\perp) = \mathcal{O}(K^{-1}).$$

Implication of asymptotic incoherence: Sample more at low Fourier frequencies where the local coherence is high, and subsample at higher Fourier frequencies.

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Notation: Given vectors $\mathbf{M} = (M_1, \dots, M_r)$ and $\mathbf{N} = (N_1, \dots, N_r)$ with $0 = M_0 < M_1 < \dots < M_r = N$ and $0 = N_0 < N_1 < \dots < N_r = N$, let

$$\mu_{\mathbf{N}, \mathbf{M}}(U)[k, l] := \sqrt{\mu(P_{N_k}^{N_{k-1}} U) \cdot \mu(P_{N_k}^{N_{k-1}} U P_{M_l}^{M_{l-1}})},$$

with $P_{j_2}^{j_1} \alpha = (0, \dots, 0, \alpha_{j_1} + 1, \dots, \alpha_{j_2}, 0, \dots, 0)$.

Main theorem (Adcock, Hansen, P & Roman, '13)

For $\epsilon > 0$ and $1 \leq k \leq r$,

$$1 \gtrsim \frac{N_k - N_{k-1}}{m_k} \cdot (\log(\epsilon^{-1}) + 1) \cdot \left(\sum_{l=1}^r \mu_{\mathbf{N}, \mathbf{M}}[k, l] \cdot s_l \right) \cdot \log(N),$$

and $m_k \gtrsim \hat{m}_k \cdot (\log(\epsilon^{-1}) + 1) \cdot \log(N)$, where \hat{m}_k satisfies

$$1 \gtrsim \sum_{k=1}^r \left(\frac{N_k - N_{k-1}}{\hat{m}_k} - 1 \right) \cdot \mu_{\mathbf{N}, \mathbf{M}}[k, l] \cdot \tilde{s}_k, \quad \forall l = 1, \dots, r,$$

$\sum_{k=1}^r \tilde{s}_k \leq s$, $\tilde{s}_k \leq \max\{\|P_{N_k}^{N_{k-1}} U \xi\|^2 : \|\xi\|_\infty = 1, \xi \text{ is } (\mathbf{M}, \mathbf{s})\text{-sparse}\}$.

Suppose that ξ is a minimizer of

$$\min_{\eta \in \mathbb{C}^N} \|\eta\|_1 \quad \text{subject to} \quad \left\| P_{\Omega_{\mathbf{N}, \mathbf{m}}} U \eta - P_{\Omega_{\mathbf{N}, \mathbf{m}}} \hat{f} \right\| \leq \delta.$$

Then, with probability exceeding $1 - s\epsilon$, we have that

$$\|\xi - x\| \leq C \cdot (\delta \cdot \sqrt{\kappa} \cdot (1 + L \cdot \sqrt{s}) + \sigma_{\mathbf{M}, \mathbf{s}}(f)),$$

for constant C , $s := \sum_{k=1}^r s_k$, $\kappa = \max_{1 \leq k \leq r} \left\{ \frac{N_k - N_{k-1}}{m_k} \right\}$, $L = 1 + \frac{\sqrt{\log(\epsilon^{-1})}}{\log(4\kappa N \sqrt{s})}$
and $\sigma_{\mathbf{M}, \mathbf{s}}(f)$ is the best ℓ^1 approximation of f by an (\mathbf{M}, \mathbf{s}) -sparse vector.

Fourier sampling and wavelet recovery

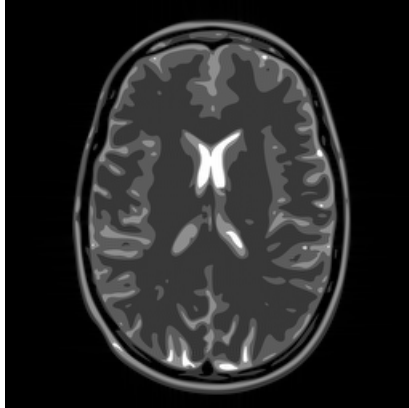
(Adcock, Hansen, P & Roman, '13)

If the sampling levels and sparsity levels correspond to wavelet scales, then the number of samples required at the k^{th} Fourier sampling level is

$$m_k \gtrsim \text{Log factors} \times (s_k + \sum_{j \neq k} s_j A^{-|j-k|})$$

where A is some constant dependent on the smoothness and the number of vanishing moments of the wavelet basis.

The test phantom



Test phantom constructed by Guerquin-Kern, Lejeune, Pruessmann, Unser, '12

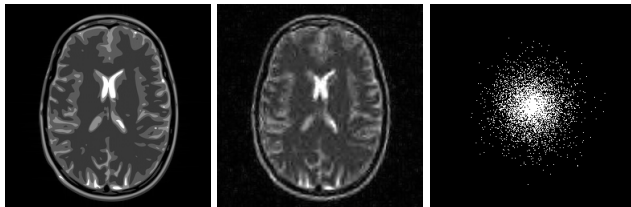
Resolution dependence (5% samples, varying resolution)

Asymptotic sparsity and asymptotic incoherence are only witnessed when N is large. Thus, multi-level sampling only reaps their benefits for large values of N and the success of compressed sensing is **resolution dependent**.

256x256

Error:

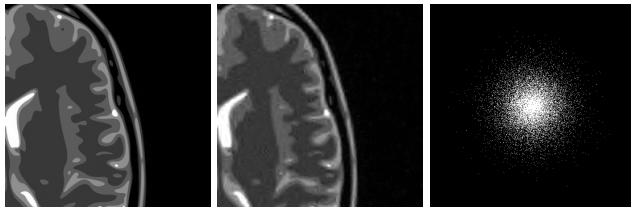
19.86%



512x512

Error:

10.69%

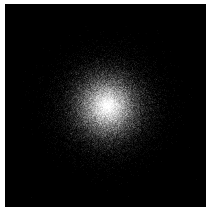
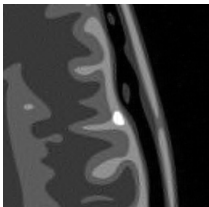
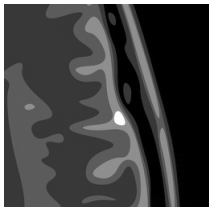


Resolution dependence (5% samples, varying resolution)

1024x1024

Error:

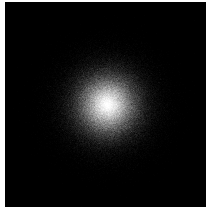
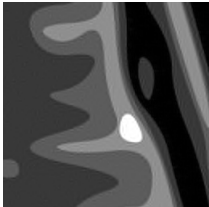
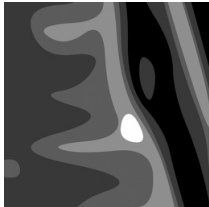
7.35%



2048x2048

Error:

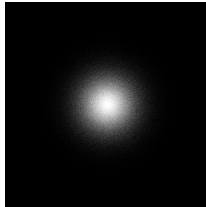
4.87%



4096x4096

Error:

3.06%



The optimal sampling strategy is signal dependent

Existing theory on compressed sensing has been based constructing **one** sampling pattern for the recovery of **all** s -sparse signals.

However, recall our conditions

$$1 \gtrsim \frac{N_k - N_{k-1}}{m_k} \cdot (\log(\epsilon^{-1}) + 1) \cdot \left(\sum_{l=1}^r \mu_{\mathbf{N}, \mathbf{M}}[k, l] \cdot s_l \right) \cdot \log(N),$$

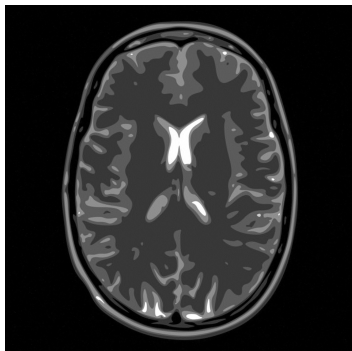
$$1 \gtrsim \sum_{k=1}^r \left(\frac{N_k - N_{k-1}}{\hat{m}_k} - 1 \right) \cdot \mu_{\mathbf{N}, \mathbf{M}}[k, l] \cdot \tilde{s}_k, \quad \forall l = 1, \dots, r,$$

Clearly, our sampling pattern should depend both on the signal structure and local incoherences.

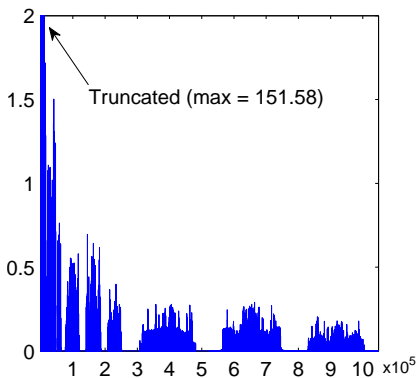
The optimal sampling strategy is signal dependent

We aim to recover wavelet coefficients x from 10% Fourier samples.

Original image (1024x1024)



Signal structure of x



The optimal sampling strategy is signal dependent

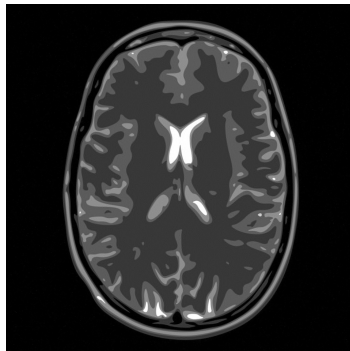
For some multi-level sampling scheme, $\Omega_{\mathbf{N},\mathbf{m}}$

By solving

$\min_{\eta \in \mathbb{C}^N} \|\eta\|_1$ subject to

$$\left\| P_{\Omega_{\mathbf{N},\mathbf{m}}} U\eta - P_{\Omega_{\mathbf{N},\mathbf{m}}} \hat{f} \right\| \leq \delta$$

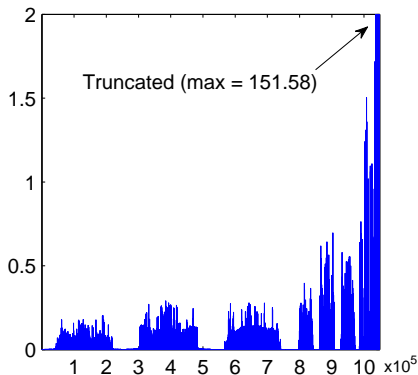
Reconstruction



The optimal sampling strategy is signal dependent

We now flip the coefficient vector x , then x_{flip} has the same sparsity of x but different signal structure.

Signal structure of x_{flip}



If only sparsity matters...

...then using the same sampling pattern $\Omega_{N,m}$

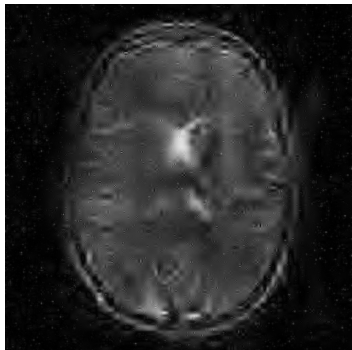
$$\min_{\eta \in \mathbb{C}^N} \|\eta\|_1 \text{ subject to}$$

$$\left\| P_{\Omega_{N,m}} U \eta - P_{\Omega_{N,m}} \hat{f}_{flip} \right\| \leq \delta$$

will recover x_{flip} . So we can recover $x = (x_{flip})_{flip}$.

The optimal sampling strategy is signal dependent

The reconstruction



So signal structure matters...

This observation of signal dependence opens up the possibility of designing optimal sampling patterns for **specific** types of signals.

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Compressed sensing with non-orthonormal systems

A more general problem

We have explained the use of multi-level sampling schemes for orthonormal systems. But...

- ▶ Our sparsifying dictionary could be a frame (e.g. shearlets), or we may want to minimize with a TV norm.
- ▶ Our sampling vectors need not form an orthonormal basis (e.g. nonuniform Fourier samples).

A more general problem

We aim to extend the theory to understand how local incoherence and local signal structures affect the minimizers of

$$\min_{\eta \in \mathbb{C}^N} \|D\eta\|_1 \quad \text{subject to} \quad \left\| P_{\Omega_{N,m}} U\eta - P_{\Omega_{N,m}} \hat{f} \right\| \leq \delta$$

where $\hat{f} = Ux$, $U \in \mathbb{C}^{M \times N}$, $D \in \mathbb{C}^{d \times N}$.

We will focus on the following two examples:

Case I: D is the finite differences matrix and U is the Fourier matrix.

Case II: $D = I$ and U is the matrix for shearlets with Fourier sampling.

Co-sparsity

(Introduced by Elad et al, '11)

$$\min_{\eta \in \mathbb{C}^N} \|D\eta\|_1 \text{ subject to } \left\| P_{\Omega_{N,m}} U\eta - P_{\Omega_{N,m}} \hat{f} \right\| \leq \delta$$

- ▶ If the rows of D are highly redundant, then Dx cannot be sparse unless $x = 0 \implies$ focus on the **zeros** of Dx .
- ▶ The co-sparse set is the index set Λ for which $P_\Lambda Dx = 0$. Let $\dim \text{Null}(P_\Lambda D) = s$.

Let $\{w_j : j = 1, \dots, s\}$ be an orthonormal basis for $\text{Null}(P_\Lambda D)$ and

$$W_\Lambda = (w_1 | w_2 | \dots | w_s),$$

Assumptions and incoherence

$$\min_{\eta \in \mathbb{C}^N} \|D\eta\|_1 \text{ subject to } \|P_{\Omega_{N,m}} U\eta - P_{\Omega_{N,m}} \hat{f}\| \leq \delta.$$

- We require that $X := W_\Lambda(W_\Lambda^* U^* U W_\Lambda)^{-1} W_\Lambda^*$ exists. (C_{inv})
- The following conditions determine the types of signals that can be recovered (cf. notions of identifiability by Fuchs '04, Tropp '04, Peyré '11):

$$\|(D^* P_\Lambda)^\dagger U^* U X W_\Lambda\|_{\infty \rightarrow \infty} < 1 \quad (C_{id1})$$

$$\inf_{u \in \text{Null}(D^* P_\Lambda)} \|(D^* P_\Lambda)^\dagger D^* P_{\Lambda^c} \text{sgn}(P_{\Lambda^c} D x) - u\|_\infty < 1. \quad (C_{id2})$$

- Coherence is between U and $\text{Null}(P_\Lambda D) = \text{span}\{w_j : j = 1, \dots, s\}$:
let

$$\mu_{N,M}[k] = \max \{ \mu_{N,M}(U W_\Lambda)[k], \mu_{N,M}(U X W_\Lambda)[k], \mu_{N,M}(U(P_\Lambda D)^\dagger)[k] \}$$

$$\mu_{N,M}[k, j] = \sqrt{\mu_{N,M}[k] \cdot \max \{ \mu_{N,M}(U W_\Lambda)[k, j], \mu_{N,M}(U X W_\Lambda)[k, j] \}}$$

Recovery statement (P.'13)

Let $\epsilon > 0$ and $\hat{f} = UX$. Recall $X := W_\Lambda(W_\Lambda^* U^* U W_\Lambda)^{-1} W_\Lambda^*$. For

$$m_k \gtrsim \log(\epsilon^{-1} + 1) \cdot \log(KN\sqrt{s}) \cdot (N_k - N_{k-1}) \cdot \sum_{j=1}^r \mu_{\mathbf{N}, \mathbf{M}}[k, j] \cdot s_j,$$

and $m_k \gtrsim \log(\epsilon^{-1} + 1) \cdot \log(KN\sqrt{s}) \cdot \hat{m}_k$ with

$$1 \gtrsim \sum_{k=1}^r \left(\frac{N_k - N_{k-1}}{\hat{m}_k} - 1 \right) \cdot \mu_{\mathbf{N}, \mathbf{M}}[k] \cdot \tilde{s}_k$$

$\sum_{k=1}^r \tilde{s}_k \leq \|UX\|^2 s$, $\tilde{s}_k \leq \max\{\|P_{N_k}^{N_{k-1}} UX W_\Lambda \xi\|^2 : \|\xi\|_\infty = 1\}$.
Suppose that ξ is a minimizer of

$$\min_{\eta \in \mathbb{C}^N} \|D\eta\|_1 \text{ subject to } \|P_{\Omega_{\mathbf{N}, \mathbf{m}}} U\eta - P_{\Omega_{\mathbf{N}, \mathbf{m}}} \hat{f}\| \leq \delta.$$

Then, with probability exceeding $(1 - s\epsilon)$,

$$\|\xi - x\| \leq C \cdot \|(P_\Lambda D)^\dagger\|_{1 \rightarrow 2} \cdot \left(\left(\sqrt{\|X\|} + \sqrt{s} \cdot L \right) \cdot \sqrt{\kappa} \cdot \delta + \|P_\Lambda D x\|_1 \right).$$

for constant C , $s := \sum_{k=1}^r s_k$ and $\kappa = \max_{1 \leq k \leq r} \frac{N_k - N_{k-1}}{m_k}$.

Case I: Total variation with Fourier samples

U is the unitary Discrete Fourier matrix,

$$D = \begin{pmatrix} -1 & +1 & & & 0 \\ & -1 & +1 & & \\ & & \ddots & \ddots & \\ 0 & & & -1 & +1 \end{pmatrix}$$

- ▶ Given Λ , if $\Lambda^c = \{\gamma_j : j = 1, \dots, s-1\}$ with $\gamma_0 = 0$, $\gamma_s = 2^n$, then

$$\text{Null}(P_\Lambda D) = \text{span} \left\{ (\gamma_j - \gamma_{j-1})^{-1/2} \chi_{(\gamma_{j-1}, \gamma_j]} : j = 1, \dots, s \right\}.$$

- ▶ (C_{inv}) holds, (C_{id1}) is trivial and (C_{id2}) holds if there is **no stair-casing**: $\nexists j$ s.t. $(P_{\Lambda^c} D x)_j = (P_{\Lambda^c} D x)_{j+1} = \pm 1$. (Peyré et al, 2011)
- ▶ $\mu_{\mathbf{N}, \mathbf{M}}[k] = \mathcal{O}\left(\frac{1}{N_{k-1}}\right)$, $\mu_{\mathbf{N}, \mathbf{M}}[k, j] = \mathcal{O}\left(\min\left\{\frac{1}{N_{k-1}}, \sqrt{\frac{L_{j-1}}{N_{k-1} 2^n}}\right\}\right)$ where L_j is the shortest length of the support of vectors in level j .

Case II: Shearlet reconstructions from Fourier samples

$$\min_{\eta \in \mathbb{C}^N} \|\eta\|_1 \quad \text{subject to} \quad \left\| P_{\Omega_{N,m}} U_{df} V_{ds}^* \eta - P_{\Omega_{N,m}} \hat{f} \right\| \leq \delta.$$

where V_{ds} is some (tight) discrete shearlet transform. Let a_j be the j^{th} row of V_{ds} .

- ▶ $\Lambda^c =: \Delta$ indexes the sparse shearlet representation. We can assume that $\{a_j : j \in \Delta\}$ is a linearly independent set. In particular, $X = P_\Delta (P_\Delta U^* U P_\Delta)^{-1} P_\Delta = P_\Delta (P_\Delta V_{ds} V_{ds}^* P_\Delta)^{-1} P_\Delta$ exists and (C_{inv}) holds.
- ▶ (C_{id2}) is trivial and (C_{id1}) is true whenever

$$\sup_{i \notin \Delta} \sum_{j \in \Delta} |\langle a_i, a_j \rangle| + \max_{i \in \Delta} \sum_{j \in \Delta, j \neq i} |\langle a_i, a_j \rangle| < 1.$$

Note that the Gram matrices of shearlets/curvelets are known to have strong off diagonal decay properties (Grohs & Kutyniok, 2012).

- ▶ $\mu(P_K^\perp U) = \mathcal{O}(K^{-1})$, $\mu(UP_K^\perp) = \mathcal{O}(K^{-1})$.

Example

$$\min_{\eta \in \mathbb{C}^N} \|\eta\|_1 \text{ subject to } \left\| P_{\Omega_{N,m}} U_{df} V_*^{-1} \eta - P_{\Omega_{N,m}} \hat{f} \right\| \leq \delta.$$

6.25% subsampling map
(2048x2048)

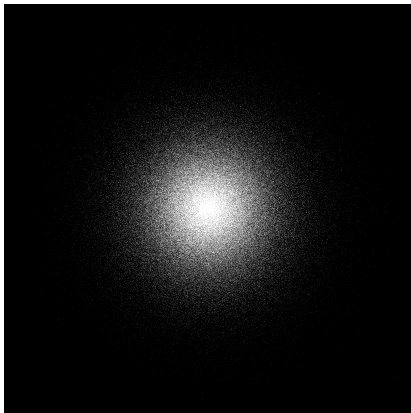


Image
(2048x2048)



Courtesy of Anders Hansen and Bogdan Roman

Example

DB-4 reconstruction
(2048x2048)



Shearlet reconstruction
(2048x2048)



Courtesy of Anders Hansen and Bogdan Roman

Example

Curvelet reconstruction
(2048x2048)



Contourlet reconstruction
(2048x2048)



Courtesy of Anders Hansen and Bogdan Roman

Conclusions

- ▶ There is a gap between the theory and the use of compressed sensing in many real world problems.
- ▶ By introducing notions of asymptotic incoherence, asymptotic sparsity and multi-level sampling, we can explain the success of variable density sampling schemes.
- ▶ Two key consequences of our theory:
 - (1) Compressed sensing is **resolution dependent**.
 - (2) Successful recovery is **signal dependent**, thus, an understanding of local incoherence and sparsity patterns of certain types of signals can lead to optimal sampling patterns.
- ▶ These ideas are applicable to non-orthonormal systems, including frames and total variation.
- ▶ *Breaking the coherence barrier: asymptotic incoherence and asymptotic sparsity in compressed sensing.* Adcock, Hansen, Poon & Roman '13

Not covered in this talk: **Extension to infinite dimensional framework**

- ▶ We recovered $x \in \mathbb{C}^N$ from Ux , $U \in \mathbb{C}^{N \times N}$. But the MRI problem samples the **continuous** Fourier transform and is **infinite dimensional**. Direct application of finite dimensional methods results in **artefacts**.