Numerical Harmonic Analysis Group

Approximate dual Gabor atoms via the adjoint lattice method

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General Aspects of Gabor Analysis

Gabor analysis is a branch of abstract harmonic analysis.

- Gabor analysis can be described over general locally compact Abelian groups
- It makes use of certain function spaces (e.g. *modulation spaces*) in order to describe boundedness of operators
- The theory of *Banach Gelfand triples* forms the backbone for Gabor analysis
- It provides criteria for the choice of good windows, or the continuous dependence of dual atoms on the lattice
- **The algorithmic aspect is solved for 1D and in progress for multi-dimensions**

General Aspects of Gabor Analysis

- **Linear Aspects**: The use of dual frame computed using the pseudo-inverse Gabor Riesz basis: biorthogonal system
- **Algebraic Aspects**: lattice (Abelian Groups) act on a Hilbert space of signals via some *projective* representation.
- **Functional Analytic features**: Convergence of double sums over lattice appear to be complicated (Bessel condition, unconditional convergence, etc.)
- Specific properties of the acting **Weyl-Heisenberg group** like phase factors.

Main Questions in Gabor Analysis

Denis Gabor, Question: Given a smooth function g which is well localized near the origin, what can one say about the pairs of lattice constants such that the family of TF-translates, usually denoted by $G(g, a, b)$ is a Gabor frame, i.e. allows to span all of the Hilbert space $\mathbf{L}^2(\mathbb{R}^d)$ in a stable way, using only $\ell^2(\mathbb{Z}^{2d})$-coefficients.

Frames, Answer: There exists $A, B > 0$ such that the family (g_λ) satisfies for all $f \in \mathbf{L}^2(\mathbb{R}^d)$

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, g_\lambda \rangle|^2 \leq B\|f\|^2 \tag{1}$$

Main Questions in Gabor Analysis II

Equivalent to the invertibility of the so-called *frame operator* S defined as

$$Sf = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda. \tag{2}$$

The fact that in the *regular case* Λ is a discrete subgroup of the additive group $G \times \hat{G}$ (resp. of *phase space*) implies that $S = S_{g,\Lambda}$ satisfies important *commutation relations*, i.e.

$$S \circ \pi(\lambda) = \pi(\lambda) \circ S, \quad \forall \lambda \in \Lambda. \tag{3}$$

The dual frame of a regular Gabor frame is generated using the *canonical dual* atom $\tilde{g} = S^{-1}(g)$, which inherits good TF-localization properties of g. (3) is also equivalent to the so-called *Janssen representation* of the Gabor frame operator.

The Janssen Criterion

For the so-called separable case, i.e. for the case that $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, there exist different methods e.g. double preconditioning.

For the non-separable case one does not have the simple *Walnut representation* anymore and it is better to directly work with the Janssen representation:

$$S_{g,\Lambda} = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} V_g g(\lambda^\circ)\pi(\lambda^\circ),$$

where Λ° is the adjoint group (a symplectic variant of the orthogonal group). Separating terms one finds that

$$Id - C_\Lambda^{-1} \cdot S_{g,\Lambda} = \sum_{\lambda^\circ \neq 0} V_g g(\lambda^\circ)\pi(\lambda^\circ),$$

The Janssen Criterion

As a consequence we obtain as a sufficient condition that for an atom g with $\|g\|_2 = 1$ one has

$$\sum_{\lambda^\circ \neq 0} |V_g g(\lambda^\circ)| < 1$$

This implies that there is a good matching between the Gabor atom g and the lattice Λ if Λ° is adapted to the shape of $|V_g(g)|$, which is a general 2D-Gauss function for the case that g is some generalized Gauss-function (e.g. $V_g g$ is a stretched and rotated 2D Gaussian).

Wexler-Raz Identity I

Theorem

Let Λ *be a lattice in* $\mathbb{R} \times \widehat{\mathbb{R}}$ *with adjoint lattice* Λ°. *Then, for* (g, γ) *in* $\mathbf{L}^2 \times \mathbf{L}^2(\mathbb{R})$ *or* $\mathbf{S}_0 \times \mathbf{S}_0{}'(\mathbb{R})$, *the following hold:*

1. *(Fundamental Identity of Gabor Analysis)*

$$\sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \langle \pi(\lambda) g, h \rangle = red(\Lambda) \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \pi(\lambda^\circ)\gamma \rangle \langle \pi(\lambda^\circ) f, h \rangle \tag{4}$$

   for all $f, h \in \mathbf{S}_0(\mathbb{R})$, *where both sides converge absolutely.*

2. *(Wexler-Raz Identity)*

$$S_{g,\gamma,\Lambda} f = red(\Lambda) S_{f,\gamma,\Lambda^\circ} g \quad in\ \mathbf{S}_0'(\mathbb{R}^d) \tag{5}$$

   for all $f \in \mathbf{S}_0(\mathbb{R})$.

Wexler-Raz Identity II

Due to the Wexler-Raz relation we can transfer the problem of computing the dual atom to the associated Gabor Riesz basis on the adjoint lattice Λ°.

Theorem

Let the pair (g, Λ) be given, with $g \in \mathbf{S}_0(\mathbb{R}), g \neq 0$, $\|g\|_2 = 1$ and Λ being the lattice. Then it generates a Gabor frame if:

$$\kappa := \sum_{\lambda^\circ \in \Lambda^\circ} |V_g g(\lambda^0)| < 2. \tag{6}$$

Fixing notations

Definition

Let $\mathbf{H}$ be a Hilbert space, $(h_i)_{i\in I}$ be a Riesz basis for $\mathbf{H}$. Let $J \subseteq \Lambda^\circ$ and $\mathbf{H}_J$ be the closed linear span of $(h_i)_{i\in J}$. We call **local approximate dual** family and we denotes it by $(\tilde{h}_i^J)_{i\in J}$ the family obtained by applying the inverted **local Gram matrix** $GR_J = \langle h_j, h_k\rangle_{j,k\in J}$ over the set J to the original Riesz sequence.

The Gram matrix is dominated by a convolution matrix over the Abelian group Λ° since:

$$|GR|_{(\lambda^\circ,\mu^\circ)} = |\langle g_{\lambda^\circ}, g_{\mu^\circ}\rangle| = |\langle g, g_{\mu^\circ-\lambda^\circ}\rangle| = |V_g g(\mu^\circ-\lambda^\circ)| \quad \forall \lambda^\circ, \mu^\circ \in \Lambda^0 \tag{7}$$

Local approximation

Corollary

Given the assumptions of Janssen criterion and $\epsilon > 0$ there exists the subset $J_0 \in \Lambda^\circ$ such that for all $J \supseteq J_0$ the following holds:

$$\left\| f - \sum_{\lambda \in \Lambda^\circ} \langle f, h_\lambda \rangle \tilde{h}_i^J \right\| \leq \epsilon \|f\|_2, \qquad \forall f \in \mathbf{L}^2$$

Proof.

For $f = \sum_{\lambda \in \Lambda^\circ} \langle f, h_\lambda \rangle \tilde{h}_i$ the error of approximation equals

$$\left\| \sum_{\lambda \in \Lambda^\circ} \langle f, h_\lambda \rangle (\tilde{h}_i - \tilde{h}_i^J) \right\| \leq C_1 \|\langle f, h_\lambda \rangle\|_{l_2} \|\tilde{h}_i - \tilde{h}_i^J\| \leq \epsilon C_2 \|f\|_2 \quad (8)$$

$\square$ $\square$

Neumann series

We obtain that

$$|GR|_J \leq |GR|_{\Lambda^\circ} = \sum_{\lambda^\circ \in \Lambda^\circ} |V_g g(\mu^\circ - \lambda^\circ)| < 2 - \epsilon_0 < 2, \forall J \subseteq \Lambda^\circ$$

and therefore $\| Id - |GR| \|_J < 1$ for each $J \subseteq \Lambda^\circ$. Thus, under the assumption of the Neumann series theorem there exist the inverse of $(Id - Id + |GR_J|) = |GR_J|$ and it can be computed by taking the powers of the Neumann series $GR_J{}^{-1} = \sum (Id - GR_J)^k$. Using the Neumann series expansion, we obtain the coordinatewise estimate in $\ell^1(\Lambda^0)$:

$$|GR_J|^{-1} = \sum_{k=0}^{\infty} |Id_J - GR_J|^k \leq \sum_{k=0}^{\infty} |Id - GR|^k \tag{9}$$

First conclusions

The Riesz basis property and choice of the atom g from the space $\mathbf{S}_0$ assure the diagonal dominance of local Gramian matrices GR_J. Therefore, for any subset $J \subset \Lambda^0$ we have that the local Gramian GR_J is invertible via the usual Neumann series, as power series $\sum_{k=0}^{\infty}(Id_J - G_J)^k$. The sequence of $GR_J{}^{-1}$ increasing in size with the J is bounded by the inverse of full absolute Gramian matrix.

Regular mask

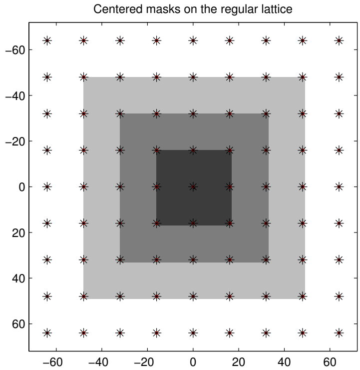

Figure: *The masks applied to the regular lattice give the areas of* $3 \times 3, 5 \times 5$ *and* 7×7 *elements around the center.*

Quincunx mask

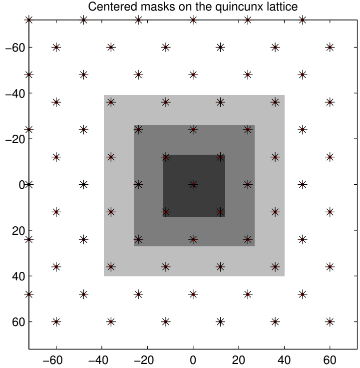

Figure: *The masks applied to the quincunx lattice give the areas of* $3 \times 3, 5 \times 5$ *and* 7×7 *elements around the center.*

Quincunx mask

We measure the error of approximation in the operator norm

$$\|Id - G \cdot T'\| < \epsilon \tag{10}$$

where we take the matrix product of the adjoint to the Gabor family $T = \mathcal{G}(h, \Lambda^\circ)$ (using the approximate dual window h^J) with the original Gabor frame G and we measure its deviation from the identity Id. Note that $Id = G \cdot T'$ for any frame T dual to the frame G.

Error of approximation regular lattice

Table: The error of approximation on regular lattice for $n = 240$ and $n = 720$.

$n = 240$			
Amount of elements	9	25	49
Red $= 1.67$	0.0150	0.0016	7.9930e-05
Red $= 2$	0.0118	8.0543e-04	5.7530e-05
Red $= 2.4$	0.0014	3.2339e-05	7.5416e-07
$n = 720$			
Red $= 2$	0.0089	4.7247e-04	2.6326e-05
Red $= 2.5$	0.0022	5.8341e-05	1.7727e-06
Red $= 3$	3.9259e-04	4.3178e-06	4.9901e-08

Error of approximation quincunx lattice

Table: The error of approximation on quincunx lattice for $n = 240$ and $n = 720$.

$n = 240$			
Amount of elements	5	13	25
Red = 1.5	0.0861	0.0071	0.0027
Red = 1.875	0.0120	5.5333e-04	3.2253e-05
Red = 2.5	0.0067	5.4801e-05	1.9370e-05
$n = 720$			
Red = 2	0.0188	0.0016	1.4186e-04
Red = 2.5	0.0031	6.5512e-05	3.6199e-06
Red = 3	0.0016	4.3224e-05	1.2659e-06

Comparison of the error performance for several redundancy

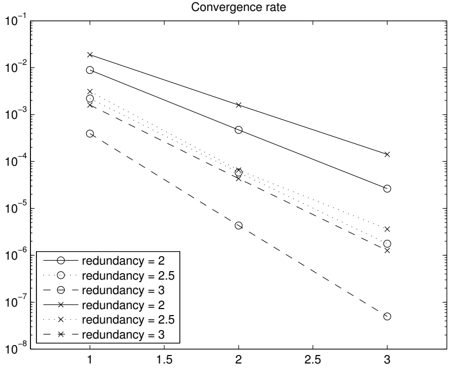

Figure: *On the horizontal axis the amount of neighbours in each direction from the center of the lattice is marked. On the vertical axis the error of approximation in the logarithmic scale is presented.*

Comparison of the approximation

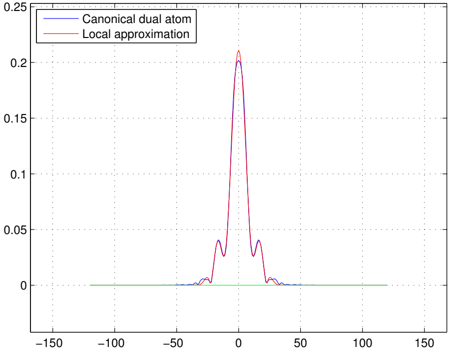

Figure: *The absolute values of the canonical dual atom and its local approximation with* 5 *elements in the case when the biggest error of approximation equal* 0.0861 *occured (see Table 1 and Table 2) i.e. for the Gabor frame with quincunx lattice and redundancy* 1.5.

Conclusions

The computational efficiency of the new approach is based on the following three principles:

- constructive discretization by periodization and sampling,
- the Wexler-Raz identity transferring the problem to the adjoint lattice
- the convergence of Neumann series corresponding to the small Gramians GR_J

Thus, the proposed procedure uses only a reduced number of time-frequency shifted atoms from the adjoint lattice to approximate the dual atom.