Nonstationary Gabor frames in audio signal processing

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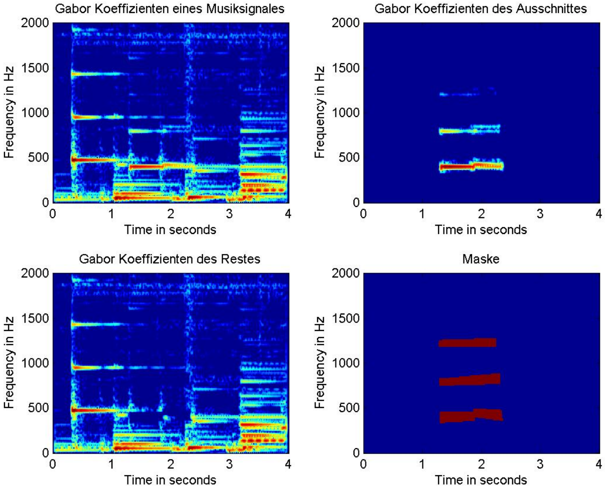
Combining mathematics and music



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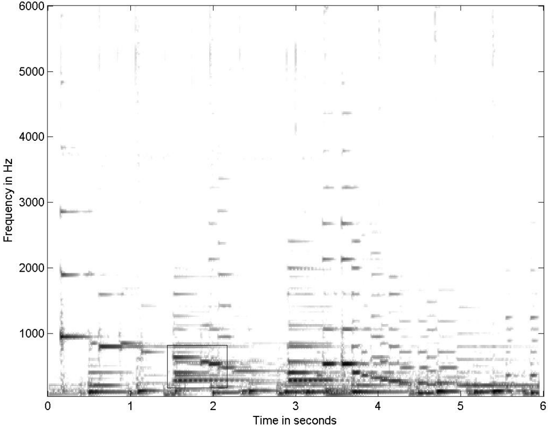

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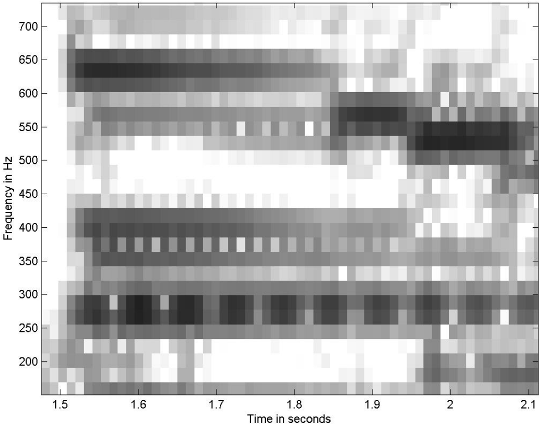

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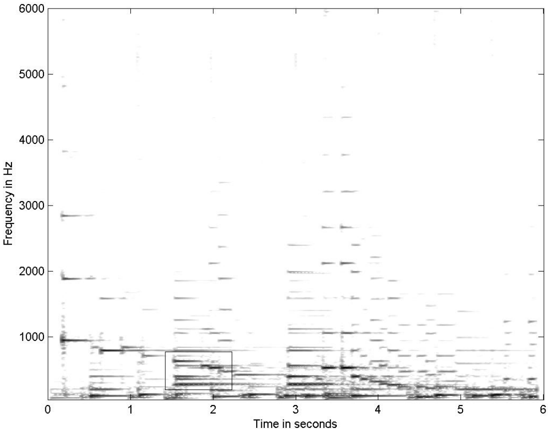

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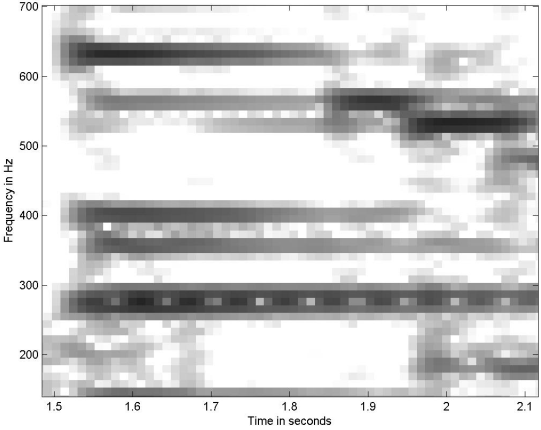

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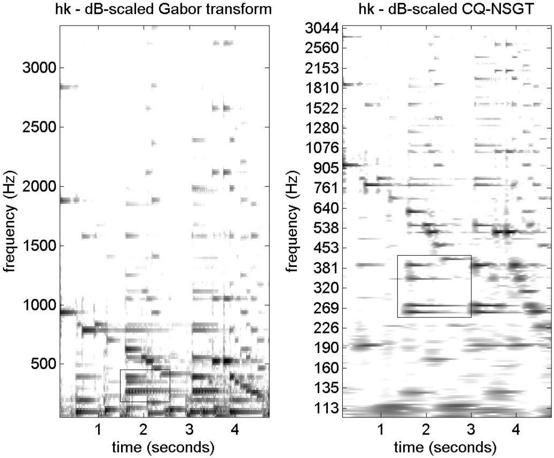

OUTLINE

- Gabor frames
- Frame operator, Walnut representation, painless case
- Nonstationary in Gabor frames
- Example applications
- Perspectives: Quilted frames

Gabor Frames

A sequence $(\psi_l)_{l\in I}$ in the Hilbert space $\mathcal{H}$ is called a *frame*, if there exist positive constants A and B:

$$A\|f\|^2 \leq \sum_{l\in I} |\langle f, \psi_l\rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}, \tag{1}$$

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Frame operator is $\mathbf{S}f = \mathbf{C}^*\mathbf{C}f = \sum_l \langle f, \psi_l \rangle \psi_l$.

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Canonical dual frame: $\tilde{\psi}_l = \mathbf{S}^{-1}\psi_l$ for all l.
Reconstruction:

$$f = \sum_l \langle f, \psi_l \rangle \tilde{\psi}_l = \sum_l \langle f, \tilde{\psi}_l \rangle \psi_l .$$

For tight frames, the frame operator reduces to $\mathbf{S} = A\mathbf{I}$, where $\mathbf{I}$ denotes the identity operator, and therefore $\mathbf{S}^{-1} = \frac{1}{A}\mathbf{I}$. The *canonical tight frame* ($\mathring{\psi}_l$) is obtained by applying $\mathbf{S}^{-\frac{1}{2}}$ to the frame elements, i.e. $\mathring{\psi}_l = \mathbf{S}^{-\frac{1}{2}}\psi_l$ for all l.

Gabor Frames

For $g \in L^2(\mathbb{R})$ (the *window*), the *short-time Fourier transform* (STFT) of $f \in L^2(\mathbb{R})$ is defined as

$$\mathcal{V}_g\left(f\right)(\tau,\omega) = \langle f, \mathbf{M}_\omega \mathbf{T}_\tau g\rangle$$

translation operator $\mathbf{T}_\tau f\left(t\right) = f\left(t-\tau\right)$
modulation operator $\mathbf{M}_\omega f\left(t\right) = f\left(t\right)e^{2\pi i\omega t}$. In other words:

$$\mathcal{V}_g\left(f\right)(\tau,\omega) = \int_{\mathbb{R}} f(t)\,\overline{g(t-\tau)}\,e^{-2\pi i\omega t}dt.$$

Gabor Frames

For a (non-zero) window function g and parameters $a, b > 0$, the set of time-frequency shifts of g

$$\mathcal{G}(g, a, b) = \{\mathbf{M}_{bm}\mathbf{T}_{an}g : m, n \in \mathbb{Z}\}$$

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(Canonical) dual frame of a Gabor frame is again a Gabor frame: generated by the *dual window* $\tilde{g} = \mathbf{S}^{-1}g$ and the same *lattice*, i.e. the set of time-frequency points $\{(an, bm) \,|\, m, n \in \mathbb{Z}\}$.

Rewriting the frame operator

If the analysis window g is (at least) in the Wiener space $W(\mathbb{R})$, then the frame operator

$$\mathbf{S}f = \sum_{m,n} \langle f, \mathbf{M}_{bm}\mathbf{T}_{an}g \rangle \mathbf{M}_{bm}\mathbf{T}_{an}g$$

can be written as

$$\mathbf{S}f(x) = \frac{1}{b}\sum_{k,n} \bar{g}(x - \frac{n}{b} - ak)g(x - ak)\mathbf{T}_{\frac{n}{b}}f(x).$$

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If g is compactly supported, and b is small enough, then $\mathbf{S}$ is diagonal! (Daubechies et al., 1988)

Frames are better than bases, but . . .

Advantages of (Gabor) frames in Music Signal Processing:

- Good time-frequency resolution is possible
- Fast processing (FFT based)
- Redundancy offers sparse representations
- Reconstruction straight-forward in painless case

but . . . what about the changing structures of signals?

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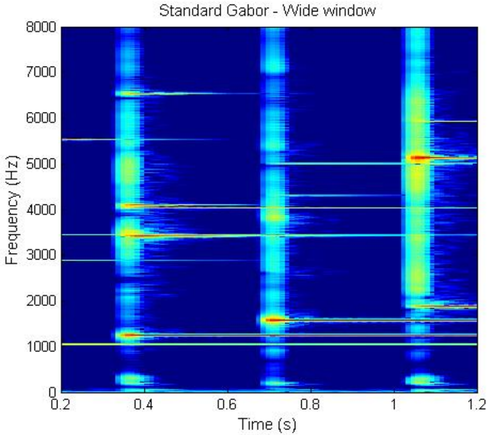

Figure: *STFT with wide window*

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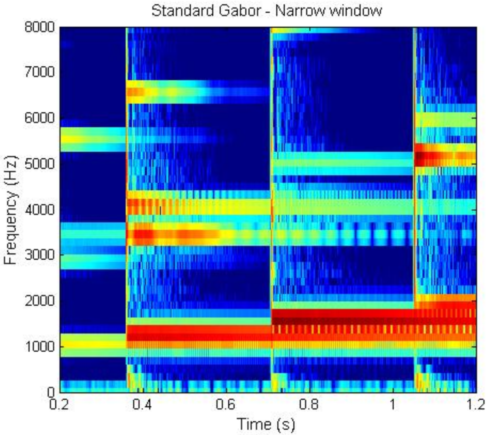

Figure: *STFT with narrow window*

The "painless" case is adaptable!

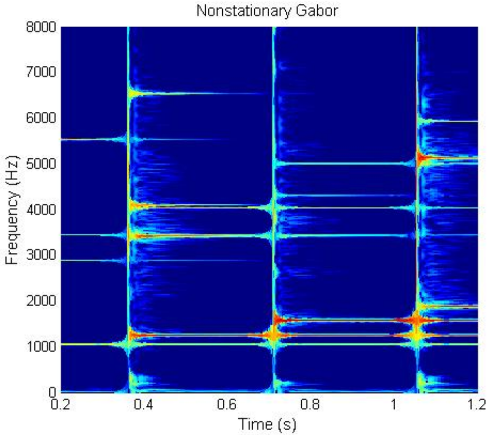

Figure: *STFT with adapted window*

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Essential ingredients:

1. Signal is localized at time- (or frequency-)positions n by multiplication with a compactly supported (or limited bandwidth, respectively) window functions g_n.
2. The Fourier transform of the localized pieces is sampled densely enough.
3. Adjacent windows overlap to avoid loss of information.
4. http://www.univie.ac.at/nonstatgab/

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For every $n \in \mathbb{Z}$, let the function $g_n \in L^2(\mathbb{R})$ be compactly supported with $\operatorname{supp}(g_n) \subseteq [c_n, d_n]$ and let b_n be chosen such that $d_n - c_n \leq \frac{1}{b_n}$. Then the frame operator

$$\mathbf{S} : f \mapsto \sum_{m,n} \langle f, g_{m,n} \rangle g_{m,n}$$

of the system

$$g_{m,n}(t) = g_n(t)\, e^{2\pi i m b_n t}, \; m, n \in \mathbb{Z},$$

is given by a multiplication operator of the form

$$\mathbf{S} f(t) = \left(\sum_n \frac{1}{b_n} |g_n(t)|^2 \right) f(t).$$

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$$h_{m,n}(t) = h_m(t - na_m). \tag{2}$$

Therefore $\widehat{h_{m,n}}(\nu) = \widehat{h_m}(\nu) \cdot e^{-2\pi i n a_m \nu}$ and the analysis coefficients may be written as

$$c_{m,n} = \langle f, h_{m,n}\rangle = \langle \hat{f}, \mathcal{F}(\mathbf{T}_{na_m} h_m)\rangle = \mathcal{F}^{-1}(\hat{f} \cdot \overline{\widehat{h_m}})(na_m).$$

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$\rightarrow$ situation analog to before, up to a Fourier transform.

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In general, the inversion of **S** can be numerically unfeasible, in the special "painless" case, the invertibility of the frame operator is easy to check and inversion is a simple multiplication:
$g_{m,n}$ forms a frame for $L^2(\mathbb{R})$ if and only if $\sum_n \frac{1}{b_n}|g_n(t)|^2 \simeq 1$.

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Canonical dual frame elements are given by:

$$\tilde{g}_{m,n}(t) = \frac{g_n(t)}{\sum_l \frac{1}{b_l}|g_l(t)|^2} e^{2\pi i m b_n t}, \tag{3}$$

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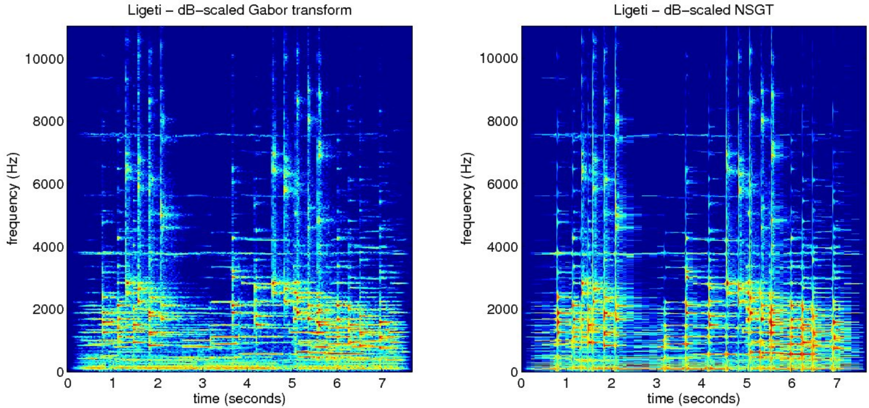

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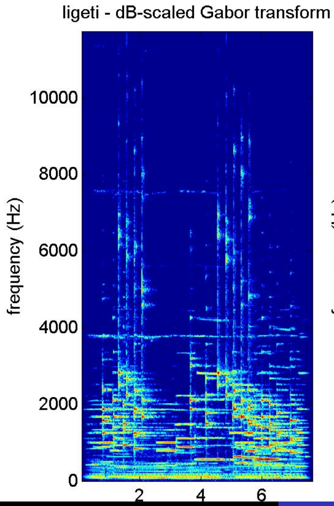
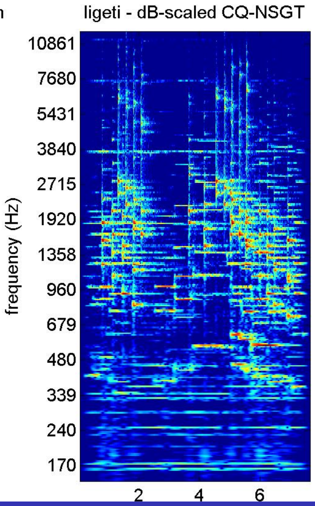
Exploring non-stationarity on the frequency side





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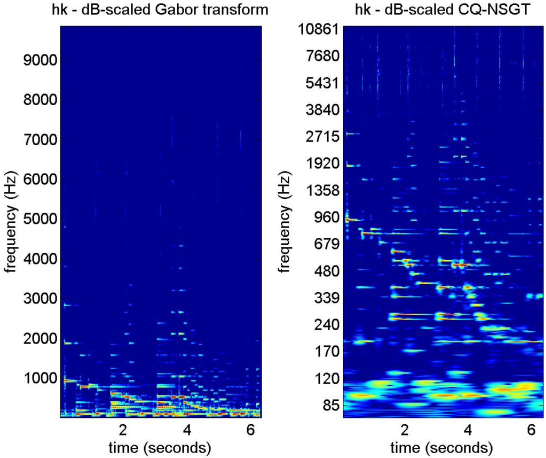

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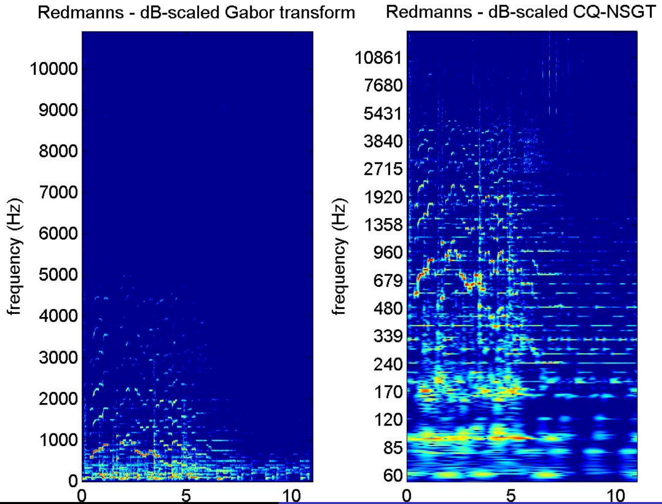


Transposition

Combining mathematics and music - the starting point

Time-frequency analysis

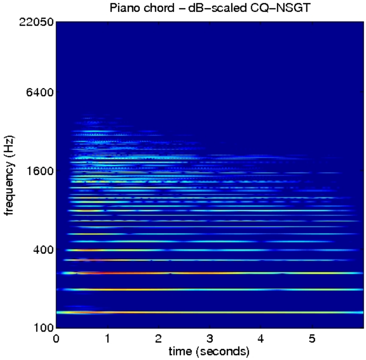
Gabor frames

The Walnut representation and the painless case

Nonstationarity in Gabor frames

Example applications

Quilted frames and local adaptation



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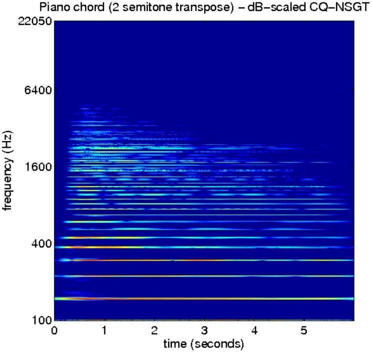
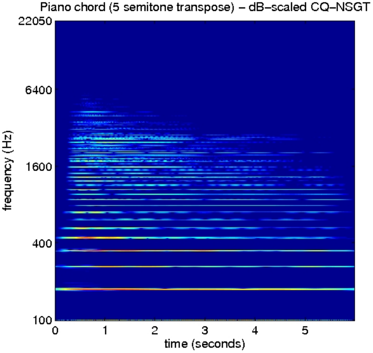
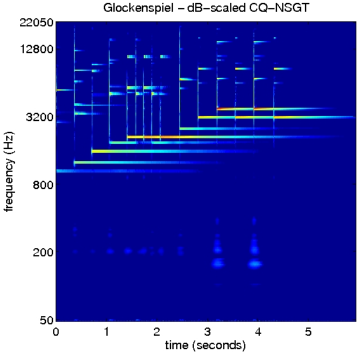
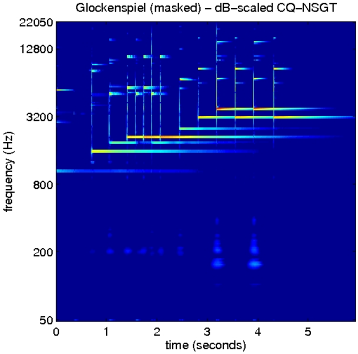


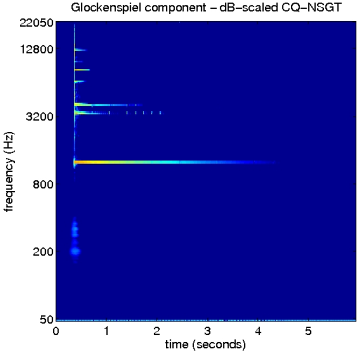


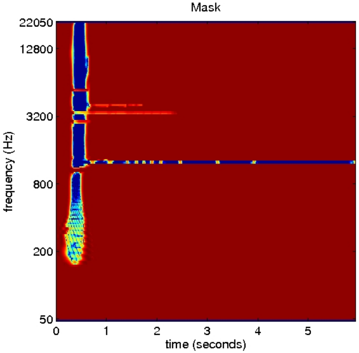
Masking


Glockenspiel – dB–scaled CQ–NSGT
frequency (Hz)
22050
12800
3200
800
200
50
0 1 2 3 4 5
time (seconds)


Glockenspiel (masked) – dB–scaled CQ–NSGT
frequency (Hz)
22050
12800
3200
800
200
50
0 1 2 3 4 5
time (seconds)

Masking


Glockenspiel component – dB–scaled CQ–NSGT
22050
12800
3200
800
200
50
frequency (Hz)
0
1
2
3
4
5
time (seconds)


Mask
22050
12800
3200
800
200
50
frequency (Hz)
0
1
2
3
4
5
time (seconds)

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References:

- P. Balazs, M. Dörfler, N. Holighaus, F. Jaillet and G. Velasco, *Theory, Implementation and Application of Nonstationary Gabor Frames* J. Comput. Appl. Math., In Press. (2011)
- Velasco, G., Holighaus, N.,Dörfler, M.,Grill, T., *Constructing an invertible constant-Q transform with non-stationary Gabor frames* Proceedings of DAFX11, Paris (2011)
  Note: constant-Q & sound patterns
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So , are we happy yet?

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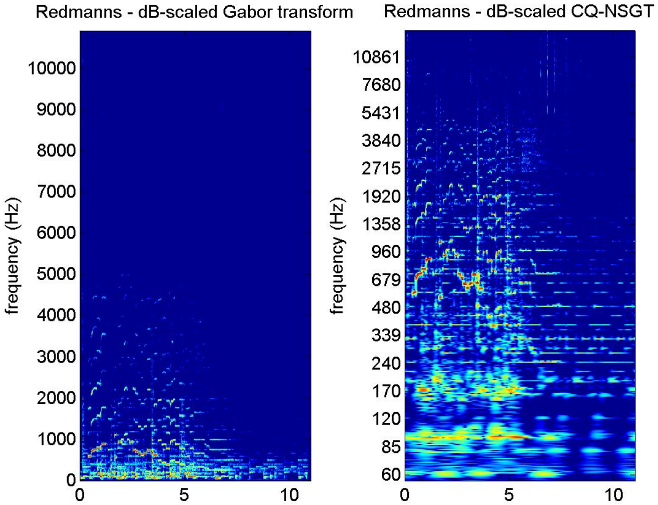

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$\rightarrow$ adaptivity in BOTH domains would be nice.
Problems: missing compactness, interpretation,..
Encouraging result: characterization by TF-shifts of operators, but NO strict locality!
Discrete version, new concept: Quilted frames

The idea and structure of quilted frames

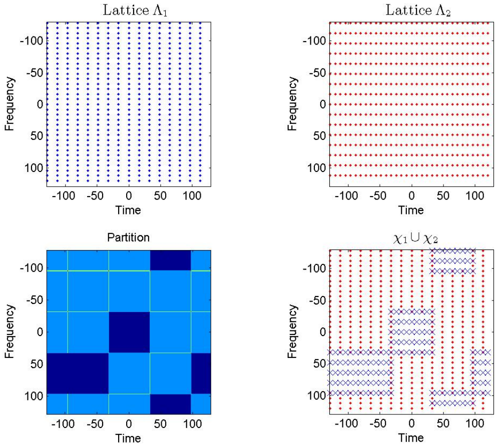

Figure: *Partition in time-frequency and resulting quilted lattice*

The idea and structure of quilted frames

Perspecitves:

- Implementation of quilted frames via ”sliced nonstationary Gabor frames”
- Adaptation via (structured) sparsity constraints
- Dictionary learning methods

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Thanks for listening!